Seeing the Möbius Transformation Like Never Before

The Theory of Gyrogroups and Gyrovector Spaces

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http://www.math.okstate.edu/mathdept/seminarlist.html

/dir_mythomas/dir_trans_tex/trans009.tex

Ahlfors' book entitled

Conformal Invariants:
Topics in Geometric Function Theory

begins with the presentation of the most general Möbius transformation of the complex open unit disc $\mathbb{D}$

$$\boxed{z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z}} \tag{1}$$

that we write as

$$\boxed{z \mapsto e^{i\theta} \frac{z_0 + z}{1 + \bar{z}_0 z} = e^{i\theta}(z_0 \oplus z)} \tag{2}$$

$$z_0 \in \mathbb{D}, \qquad \theta \in \mathbb{R}.$$

Suggestively, we define the Möbius addition $\oplus$ in the disc, allowing the generic Möbius transformation of the disc to be viewed as a Möbius *left gyrotranslation*

$$\boxed{z \mapsto z_0 \oplus z = \frac{z_0 + z}{1 + \bar{z}_0 z}} \tag{3}$$

followed by a rotation. The prefix "gyro" that we use to emphasize analogies with classical notions will soon become clear. The resulting Möbius addition in the disc $\mathbb{D}$ is neither commutative not associative. To "repair" the breakdown of commutativity in the Möbius addition we define the *gyration* (or, rotation)

$$\boxed{\mathrm{gyr}[a, b] = \frac{a \oplus b}{b \oplus a} = \frac{1 + a\bar{b}}{1 + b\bar{a}}} \tag{4}$$

giving rise to the gyrocommutative law of Möbius addition,

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a) \qquad \text{Gyrocommutative Law}$$

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$a \oplus b = \boxed{\mathrm{gyr}[a, b]}(b \oplus a)$ Gyrocommutative Law

Coincidentally, the gyration that repairs the breakdown of commutativity in the Möbius addition repairs the breakdown of associativity as well, giving rise to identities that capture analogies,

$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$	Left Gyroass. Law
$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c)$	Right Gyroa. Law
$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$	Left Loop Property
$\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a]$	Right Loop Prop.

$a \oplus (b \oplus c) = (a \oplus b) \oplus \boxed{\mathrm{gyr}[a, b]}c$	Left Gyroass. Law
$(a \oplus b) \oplus c = a \oplus (b \oplus \boxed{\mathrm{gyr}[b, a]}c)$	Right Gyroa. Law

The emerging coincidences expose an algebraic structure that merits extension by abstraction, leading to the concept of the grouplike structure called a gyrogroup.

Gyrogroups are generalized groups that share remarkable analogies with groups. In full analogy with groups

(1) gyrogroups are classified into gyrocommutative gyrogroups and non-gyrocommutative gyrogroups; and

(2) some gyrocommutative gyrogroups admit scalar multiplication, turning them into gyrovector spaces.

(3) Gyrovector spaces, in turn, provide the setting for hyperbolic geometry just as vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified.

Definition 1 (Gyrogroups). *The groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying for all $a, b, z \in G$*

$0\oplus a = a$	*Left Identity*
$\ominus a\oplus a = 0$	*Left Inverse*
$a\oplus(b\oplus z) = (a\oplus b)\oplus \mathrm{gyr}[a, b]z$	*Left Gyroassociative*
$\mathrm{gyr}[a, b] \in Aut(G, \oplus)$	*Gyroautomorphism*
$\mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b]$	*Left Loop Property*

Definition 2 (Gyrocommutative Gyrogroups). *The gyrogroup $(G, \oplus)$ is gyrocommutative if for all $a, b \in G$*

$a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$	*Gyrocommutative Law*

GYRATIONS

The Left Gyroassociative Law

$$a\oplus(b\oplus z) = (a\oplus b)\oplus\text{gyr}[a, b]z \tag{5}$$

Solving for $\text{gyr}[a, b]z$,

$$\boxed{\text{gyr}[a, b]z = \ominus(a\oplus b)\oplus\{a\oplus(b\oplus z)\}} \tag{6}$$

$\text{gyr}[a, b]$ is an automorphism of the groupoid

$$(G, \oplus)$$

called the

Gyroautomorphism generated by a and b.

The Möbius addition $\oplus_{_M}$

in $(\mathbb{V}_c, \oplus_{_M})$

$$\boxed{\mathbf{u}\oplus_{_M}\mathbf{v} = \frac{(1+\frac{2}{c^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{c^2}\|\mathbf{v}\|^2)\mathbf{u}+(1-\frac{1}{c^2}\|\mathbf{u}\|^2)\mathbf{v}}{1+\frac{2}{c^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{c^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \tag{7}$$

for all $\mathbf{u}$ and $\mathbf{v}$ in the c-ball $\mathbb{V}_c$

$$\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\} \tag{8}$$

of any real inner product space $\mathbb{V}$, $\gamma_{\mathbf{u}}$ being the Lorentz factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\frac{\|\mathbf{u}\|^2}{c^2}}} \tag{9}$$

Motivating the Möbius Vector Addition

Identifying vectors in $\mathbb{R}^2$ with complex numbers in the usual way we have

$$\mathbf{u} = (u_1, u_2) = u_1 + iu_2 = u \tag{10}$$

The inner product and the norm in $\mathbb{R}^2$ then become the real numbers

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} = Re(\bar{u}v) = \frac{\bar{u}v + u\bar{v}}{2} \\ \|\mathbf{u}\| = |u| \end{aligned} \tag{11}$$

where $\bar{u}$ is the complex conjugate of u.

Under the translation (11) of elements of the disc $\mathbb{R}^2_{c=1}$ of $\mathbb{R}^2$ into elements of the complex unit disc $\mathbb{D}$, the Möbius addition (7), with $c = 1$ for simplicity, takes the form

$$
\begin{aligned}
\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} &= \frac{(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&= \frac{(1+\bar{u}v+u\bar{v}+|v|^2)u+(1-|u|^2)v}{1+\bar{u}v+u\bar{v}+|u|^2|v|^2} \\
&= \frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})} \\
&= \frac{u+v}{1+\bar{u}v} \\
&= u\oplus_{\mathrm{M}}v
\end{aligned}
\tag{12}
$$

for all $\mathbf{u},\mathbf{v}\in\mathbb{R}^2_{c=1}$, and all $u,v\in\mathbb{D}$.

The Algebra of
Möbius Transformations
and their
Artful Applications
are found in the 2002 book

INDRA'S PEARLS
The Vision of Felix Klein

by
David Mumford
Caroline Series
David Wright

About 150 years have passed since
August Ferdinand Möbius
first studied (1855) the transformations
that now bear his name.

Yet, the rich structure he thereby exposed
is still far from being exhausted.

The Einstein addition $\oplus_{\text{E}}$

in $(\mathbb{V}_c, \oplus_{\text{E}})$

$$\boxed{\mathbf{u}\oplus_{\text{E}}\mathbf{v} = \frac{1}{1+\dfrac{\mathbf{u}\cdot\mathbf{v}}{c^2}}\left\{\mathbf{u}+\frac{1}{\gamma_{\mathbf{u}}}\mathbf{v}+\frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u}\right\}} \tag{13}$$

for all $\mathbf{u}$ and $\mathbf{v}$ in the c-ball $\mathbb{V}_c$

$$\mathbb{V}_c = \{\mathbf{v}\in\mathbb{V} : \|\mathbf{v}\| < c\} \tag{14}$$

of any real inner product space $\mathbb{V}$, $\gamma_{\mathbf{u}}$ being the Lorentz factor

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1-\dfrac{\|\mathbf{u}\|^2}{c^2}}} \tag{15}$$

Both Einstein's addition and Möbius' addition are important in modern physics. Gyro-theoretic techniques uncover the reason:

They are gyro-isomorphic to each other.

The Isomorphism Between
The Einstein Addition $\oplus_{\text{E}}$
and
The Möbius Addition $\oplus_{\text{M}}$

$$\boxed{\mathbf{u} \oplus_{\text{M}} \mathbf{v} = \frac{1}{2} \otimes (2 \otimes \mathbf{u} \oplus_{\text{E}} 2 \otimes \mathbf{v})} \tag{16}$$

$$\boxed{\mathbf{u} \oplus_{\text{E}} \mathbf{v} = 2 \otimes (\frac{1}{2} \otimes \mathbf{u} \oplus_{\text{M}} \frac{1}{2} \otimes \mathbf{v})} \tag{17}$$

Scalar Multiplication $\otimes$

for Möbius and Einstein's gyrogroups $(\mathbb{V}_c, \oplus)$

$$\boxed{\begin{aligned} r\otimes\mathbf{v} &= c\frac{(1+\|\mathbf{v}\|/c)^r - (1-\|\mathbf{v}\|/c)^r}{(1+\|\mathbf{v}\|/c)^r + (1-\|\mathbf{v}\|/c)^r}\frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= c\,tanh\left(r\,tanh^{-1}\frac{\|\mathbf{v}\|}{c^2}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} \end{aligned}} \tag{18}$$

where $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{V}_c$, $\mathbf{v} \neq \mathbf{0}$; and $r\otimes\mathbf{0} = \mathbf{0}$.

The scalar multiplication possesses the following properties. For any positive integer n and for all $r, r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{V}_c$,

$$
\begin{aligned}
n \otimes \mathbf{v} &= \mathbf{v} \oplus \ldots \oplus \mathbf{v} && \text{n terms} \\
(r_1 + r_2) \otimes \mathbf{v} &= r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v} \\
(r_1 r_2) \otimes \mathbf{v} &= r_1 \otimes (r_2 \otimes \mathbf{v}) \\
r \otimes (r_1 \otimes \mathbf{v} \oplus r_2 \otimes \mathbf{v}) &= r \otimes (r_1 \otimes \mathbf{v}) \oplus r \otimes (r_2 \otimes \mathbf{v}) \\
\| r \otimes \mathbf{v} \| &= |r| \otimes \| \mathbf{v} \| \\
\frac{|r| \otimes \mathbf{v}}{\| r \otimes \mathbf{v} \|} &= \frac{\mathbf{v}}{\| \mathbf{v} \|}
\end{aligned}
\tag{19}
$$

Remarkably, we have a *monodistributive law* but no distributive law. In general,

$$
r \otimes (\mathbf{u} \oplus \mathbf{v}) \neq r \otimes \mathbf{u} \oplus r \otimes \mathbf{v} \tag{20}
$$

Solving the Equation

$$\mathbf{a}\oplus\mathbf{x} = \mathbf{b} \tag{21}$$

for x in a gyrogroup $(G, \oplus)$. If $\mathbf{x}$ is a solution, then by the right gyroassociative law and the identity $\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{a}] = id$ we have

$$\begin{aligned} \mathbf{x} &= \mathbf{0}\oplus\mathbf{x} \\ &= (\ominus\mathbf{a}\oplus\mathbf{a})\oplus\mathbf{x} \\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a}, \ominus\mathbf{a}]\mathbf{x}) \\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathbf{x}) \\ &= \ominus\mathbf{a}\oplus\mathbf{b} \end{aligned} \tag{22}$$

Thus if a solution exists, it must be given uniquely by

$$\mathbf{x} = \ominus\mathbf{a}\oplus\mathbf{b} \tag{23}$$

Solving the Equation

$$\mathbf{x}\oplus\mathbf{a} = \mathbf{b} \tag{24}$$

for x in a gyrogroup $(G, \oplus)$. If $\mathbf{x}$ is a solution then, by the left gyroassociative law and the left loop property we have

$$\begin{aligned}
\mathbf{x} &= \mathbf{x}\oplus\mathbf{0} \\
&= \mathbf{x}\oplus(\mathbf{a}\oplus(\ominus\mathbf{a})) \\
&= (\mathbf{x}\oplus\mathbf{a})\oplus\mathrm{gyr}[\mathbf{x},\mathbf{a}](\ominus\mathbf{a}) \\
&= (\mathbf{x}\oplus\mathbf{a})\oplus(\ominus\mathrm{gyr}[\mathbf{x},\mathbf{a}]\mathbf{a}) \\
&= (\mathbf{x}\oplus\mathbf{a})\ominus\mathrm{gyr}[\mathbf{x},\mathbf{a}]\mathbf{a} \\
&= \mathbf{b}\ominus\mathrm{gyr}[\mathbf{x},\mathbf{a}]\mathbf{a} \\
&= \mathbf{b}\ominus\mathrm{gyr}[\mathbf{x}\oplus\mathbf{a},\mathbf{a}]\mathbf{a} \\
&= \mathbf{b}\ominus\mathrm{gyr}[\mathbf{b},\mathbf{a}]\mathbf{a} \\
&= \mathbf{b}\boxminus\mathbf{a}
\end{aligned} \tag{25}$$

where we use the obvious notation:

$$\mathbf{a}\ominus\mathbf{b} = \mathbf{a}\oplus(\ominus\mathbf{b}).$$

Definition 3 (Gyrogroup Dual Operation). *Let $(G, \oplus)$ be a gyrogroup. A secondary binary operation $\boxplus$ in G is defined by the equation*

$$a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b \qquad \textit{Secondary Operation}$$

Theorem 4. *Let $(G, \oplus)$ be a gyrogroup and let $\boxplus$ be its dual binary operation given by Definition (3),*

$$\boxed{a \boxplus b = a \oplus \mathrm{gyr}[a, \ominus b]b} \tag{26}$$

Then

$$\boxed{a \oplus b = a \boxplus \mathrm{gyr}[a, b]b} \tag{27}$$

and

$$Aut(G, \boxplus) = Aut(G, \oplus) \tag{28}$$

Theorem 5 (Gyrogroup Cancellation Laws). *Let $(G, \oplus)$ be a gyrogroup, and let $\boxplus$ be its dual operation. Then, for all $a, b, c \in G$,*

$$a \oplus (\ominus a \oplus b) = b \qquad \textit{Left Cancellation Law}$$
$$(b \boxminus a) \oplus a = b \qquad \textit{Right Cancellation Law}$$
$$(b \ominus a) \boxplus a = b \qquad \textit{Dual Right Cancellation Law}$$

Euclidean Geodesics

$$\begin{aligned} L_p &= \mathbf{a} + (-\mathbf{a} + \mathbf{b})t \\ L_s &= (\mathbf{b} - \mathbf{a})t + \mathbf{a} \end{aligned} \tag{29}$$

Newtonian Mechanics: $\boxed{\mathbf{v}_0 + \mathbf{a}t \quad \text{or} \quad \mathbf{a}t + \mathbf{v}_0}$

Relative to the Euclidean Metric

$$\begin{aligned} d(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} - \mathbf{b}\| \\ d(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} - \mathbf{b}\| \end{aligned} \tag{30}$$

Hyperbolic Geodesics

$$\begin{aligned} L_p &= \mathbf{a}{\oplus}({\ominus}\mathbf{a}{\oplus}\mathbf{b}){\otimes}t \\ L_s &= (\mathbf{b} \boxminus \mathbf{a}){\otimes}t{\oplus}\mathbf{a} \end{aligned} \tag{31}$$

Einsteinian Mechanics: $\boxed{\mathbf{v}_0{\oplus}\mathbf{a}{\otimes}t \quad \text{or} \quad \mathbf{a}{\otimes}t{\oplus}\mathbf{v}_0}$

Relative to the Hyperbolic Dual Metrics

$$\begin{aligned} d_{\oplus}(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a}{\ominus}\mathbf{b}\| \\ d_{\boxplus}(\mathbf{a}, \mathbf{b}) &= \|\mathbf{a} \boxminus \mathbf{b}\| \end{aligned} \tag{32}$$

$\boxed{\text{Möbius gyrovector spaces}}$
form the setting
for
the $\boxed{\text{Poincaré ball model}}$
of
hyperbolic geometry

just as

vector spaces
form the setting
for
the Euclidean model
of
Euclidean geometry.

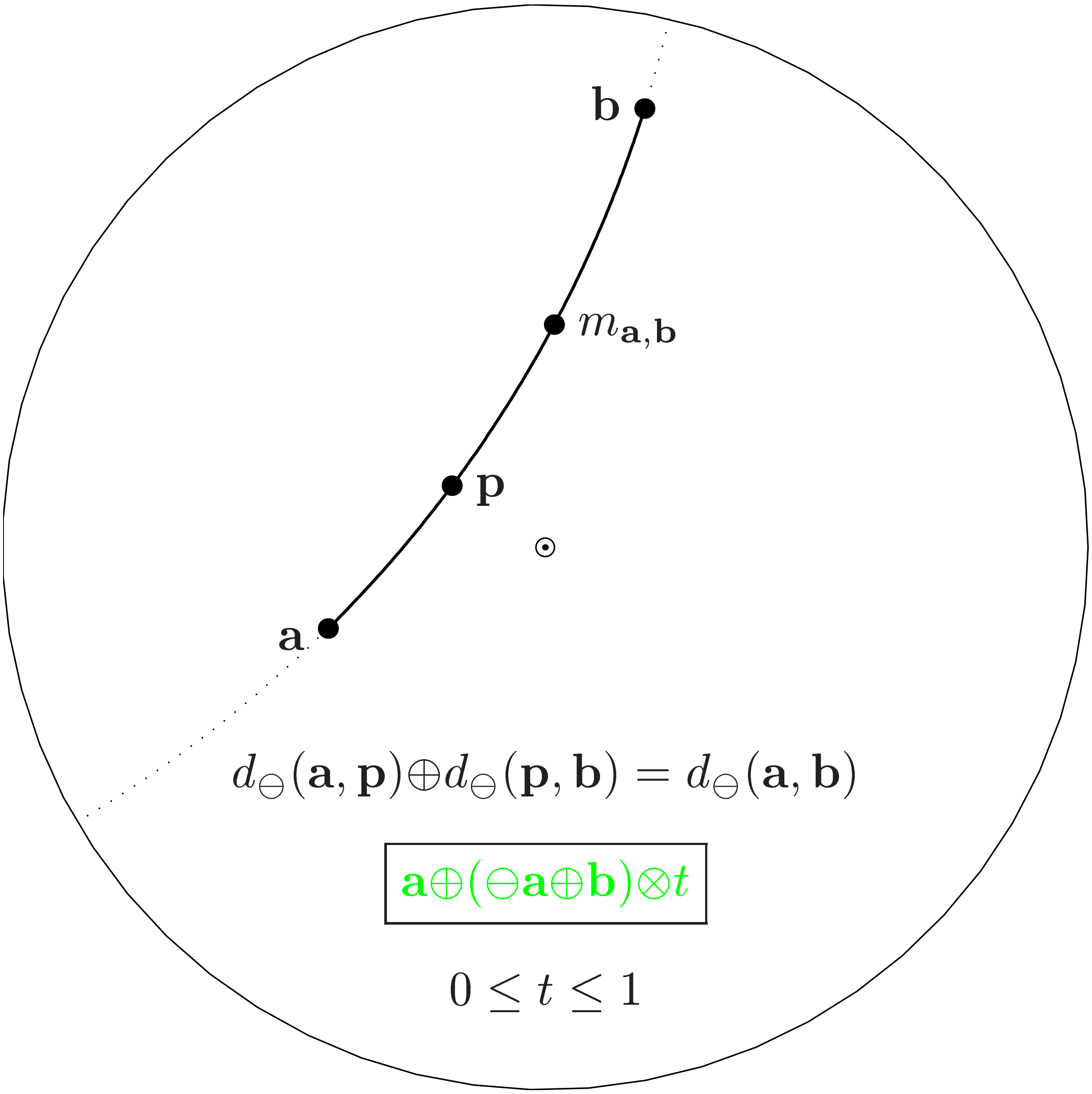

Figure 1: $\oplus = \oplus_{\text{M}}$. The gyroline segment linking the two points $\mathbf{a}$ and $\mathbf{b}$ in a Möbius gyrovector space $(\mathbb{V}_c, \oplus_{\text{M}}, \otimes)$. $\mathbf{p}$ is a generic point between $\mathbf{a}$ and $\mathbf{b}$ and $m_{\mathbf{a},\mathbf{b}}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$.

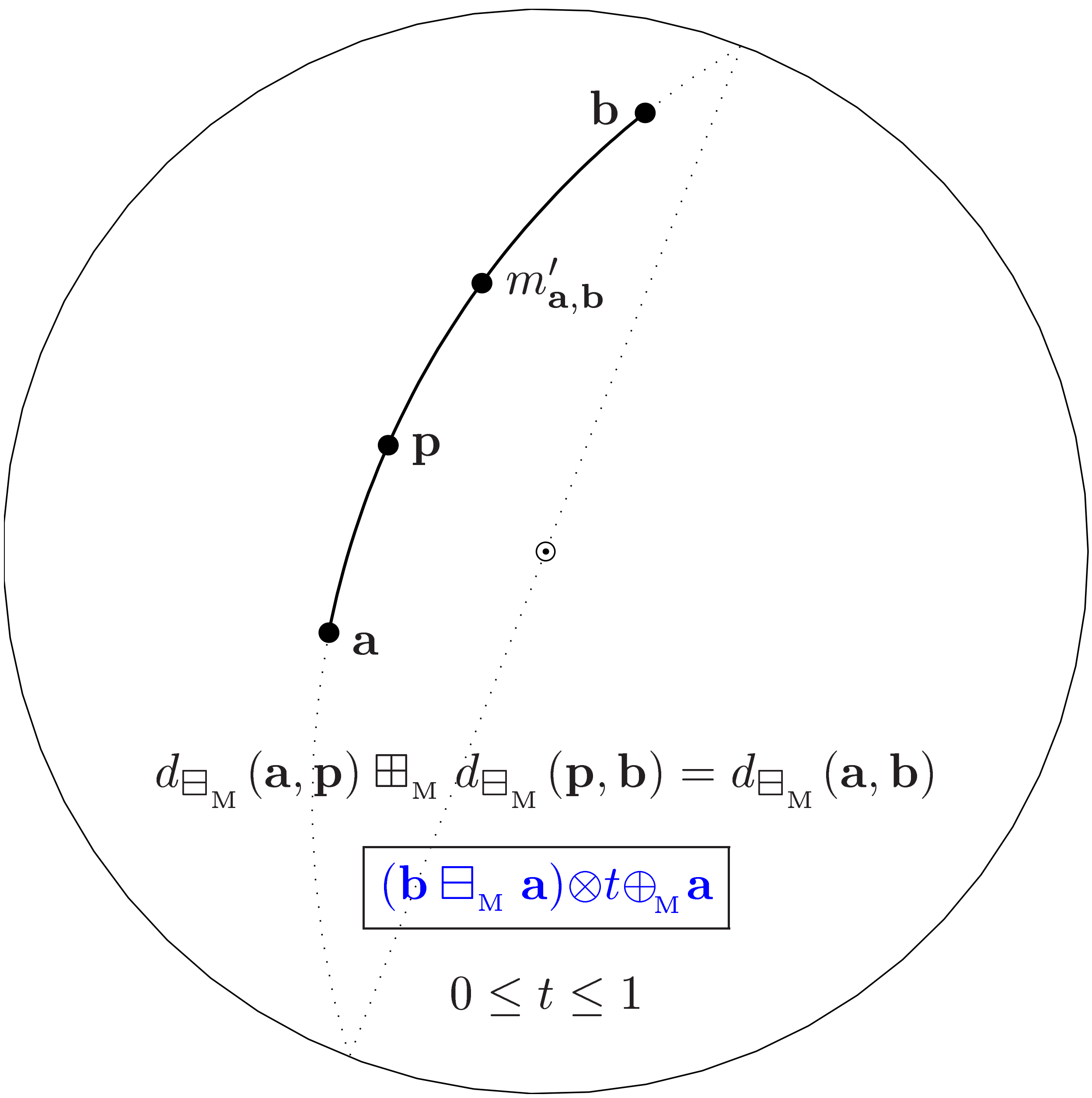

Figure 2: $\oplus = \oplus_M$. The cogyroline segment linking the two points $\mathbf{a}$ and $\mathbf{b}$ in a Möbius gyrovector space $(\mathbb{V}_c, \oplus_M, \otimes)$. $\mathbf{p}$ is a generic point cobetween $\mathbf{a}$ and $\mathbf{b}$ and $m'_{\mathbf{a},\mathbf{b}}$ is the comidpoint of $\mathbf{a}$ and $\mathbf{b}$.

$\boxed{\text{Einstein gyrovector spaces}}$

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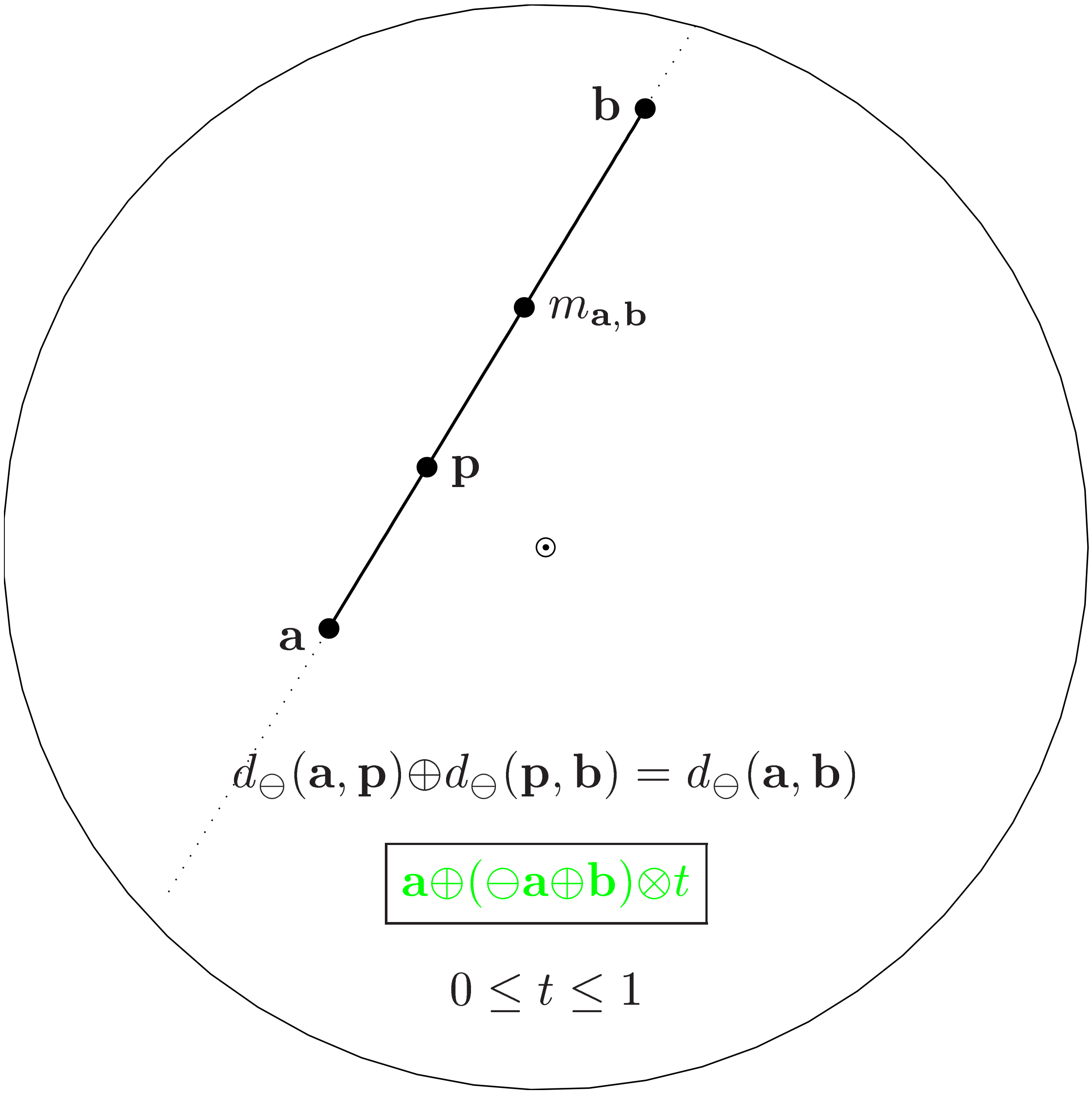


Figure 3: $\oplus = \oplus_{\text{E}}$. The gyroline segment linking the two points $\mathbf{a}$ and $\mathbf{b}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus_{\text{E}}, \otimes)$. $\mathbf{p}$ is a generic point between $\mathbf{a}$ and $\mathbf{b}$ and $m_{\mathbf{a},\mathbf{b}}$ is the midpoint of $\mathbf{a}$ and $\mathbf{b}$.

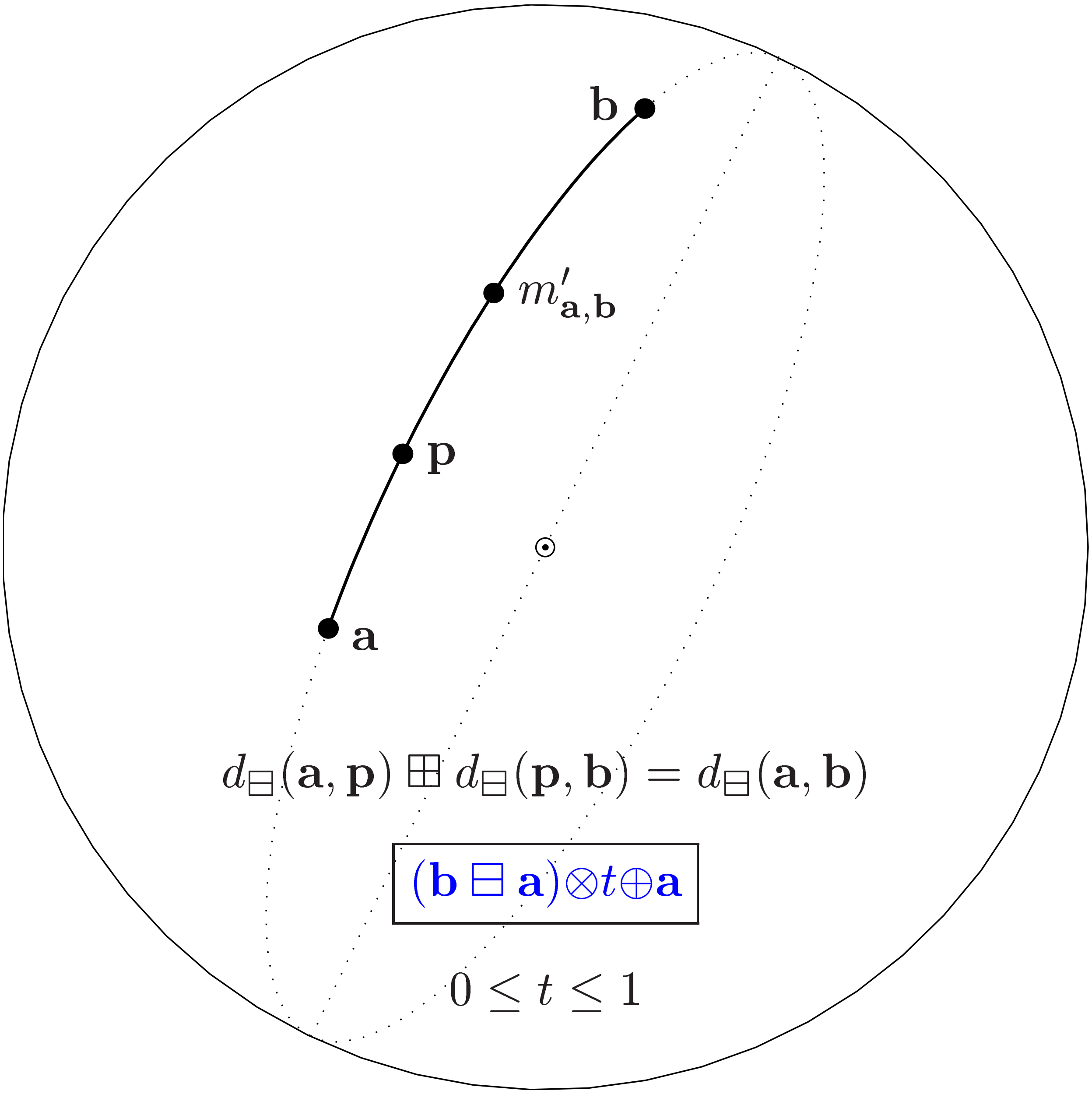


Figure 4: $\oplus = \oplus_{_E}$. The cogyroline segment linking the two points $\mathbf{a}$ and $\mathbf{b}$ in an Einstein gyrovector space $(\mathbb{V}_c, \oplus_{_E}, \otimes)$. $\mathbf{p}$ is a generaic point cobetween $\mathbf{a}$ and $\mathbf{b}$ and $m'_{\mathbf{a},\mathbf{b}}$ is the comidpoint of $\mathbf{a}$ and $\mathbf{b}$.

The dual gyrotriangle inequality

$$\|\mathbf{u}\boxminus\mathbf{v}\|\boxplus\|\mathbf{v}\boxminus\mathbf{w}\| \geq \|\mathbf{u}\boxminus\mathrm{gyr}[\mathbf{u}\boxminus\mathbf{v}, \mathbf{v}\boxminus\mathbf{w}]\mathbf{w}\| \tag{33}$$

$$\|\mathbf{u}\boxminus\mathbf{v}\|\boxplus\|\mathbf{v}\boxminus\mathbf{w}\| \geq \|\mathbf{u}\boxminus\boxed{\mathrm{gyr}[\mathbf{u}\boxminus\mathbf{v}, \mathbf{v}\boxminus\mathbf{w}]}\mathbf{w}\| \tag{34}$$

Theorem 6 (Gyration Gyrotransitive Laws). *Let L_p and L_s be dual geodesics,* (31), *in a gyrovector space* $(\mathbb{V}_c, \oplus, \otimes)$.

Let $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ *be a set of* n *points lying on* L_p *(in any order), and*

let $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ *be a set of* n *points lying on* L_s *(in any order). Then we have the gyration gyrotransitive law for geodesics,*

$$\begin{aligned}\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_3]\cdots\mathrm{gyr}[\mathbf{a}_{n-1}, \ominus\mathbf{a}_n] \quad (35)\\ = \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_n] \quad (36)\end{aligned}$$

and the dual gyration gyrotransitive law for dual geodesics,

$$\begin{aligned}\mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_2]\mathrm{gyr}[\mathbf{b}_2, \mathbf{b}_3]\cdots\mathrm{gyr}[\mathbf{b}_{n-1}, \mathbf{b}_n] \quad (37)\\ = \mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_n] \quad (38)\end{aligned}$$

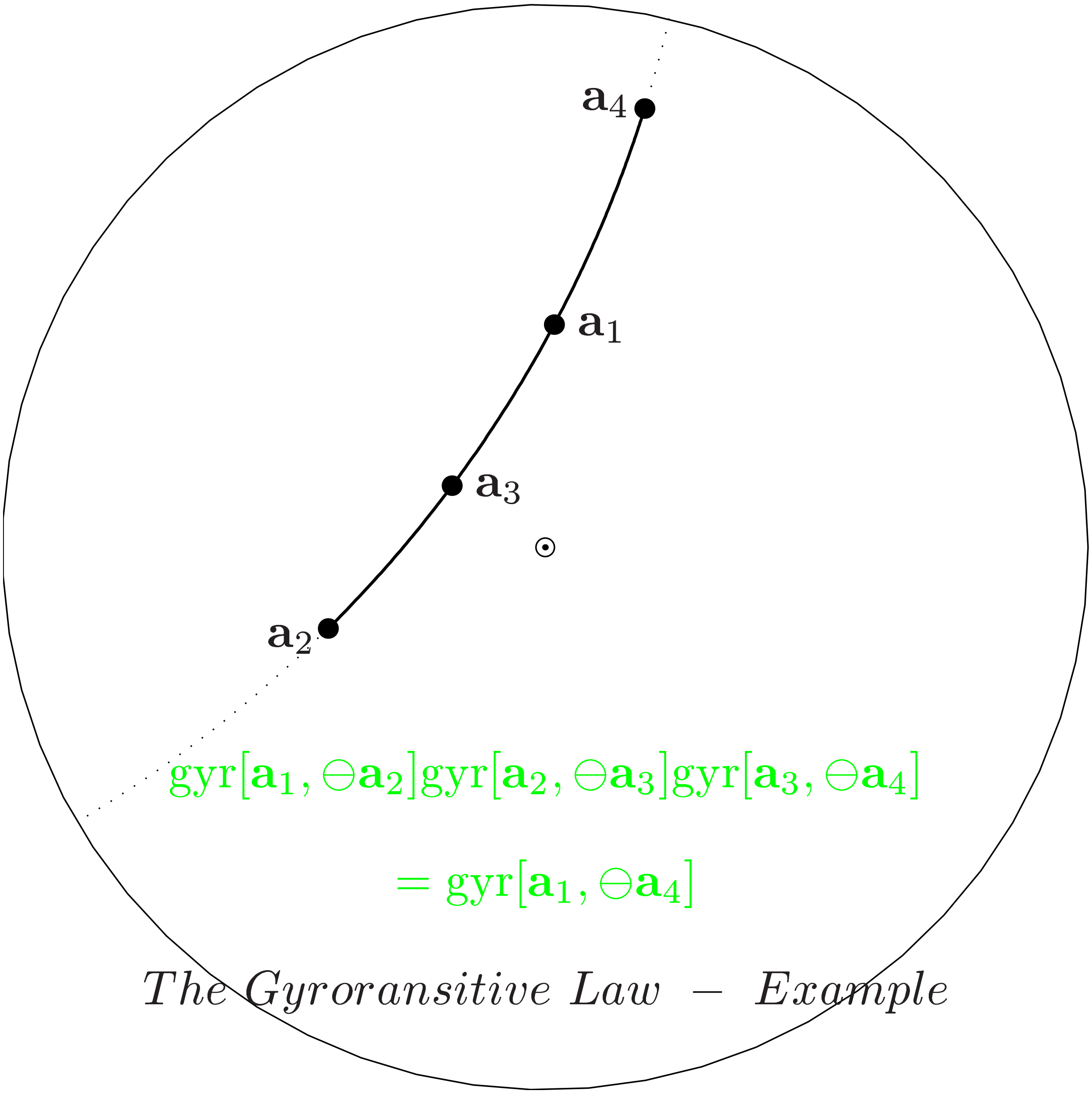

$\mathbf{a}_4$
$\mathbf{a}_1$
$\mathbf{a}_3$
$\mathbf{a}_2$
$\mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_2]\mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_3]\mathrm{gyr}[\mathbf{a}_3, \ominus\mathbf{a}_4]$
$= \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_4]$
The Gyroransitive Law – Example

Figure 5:

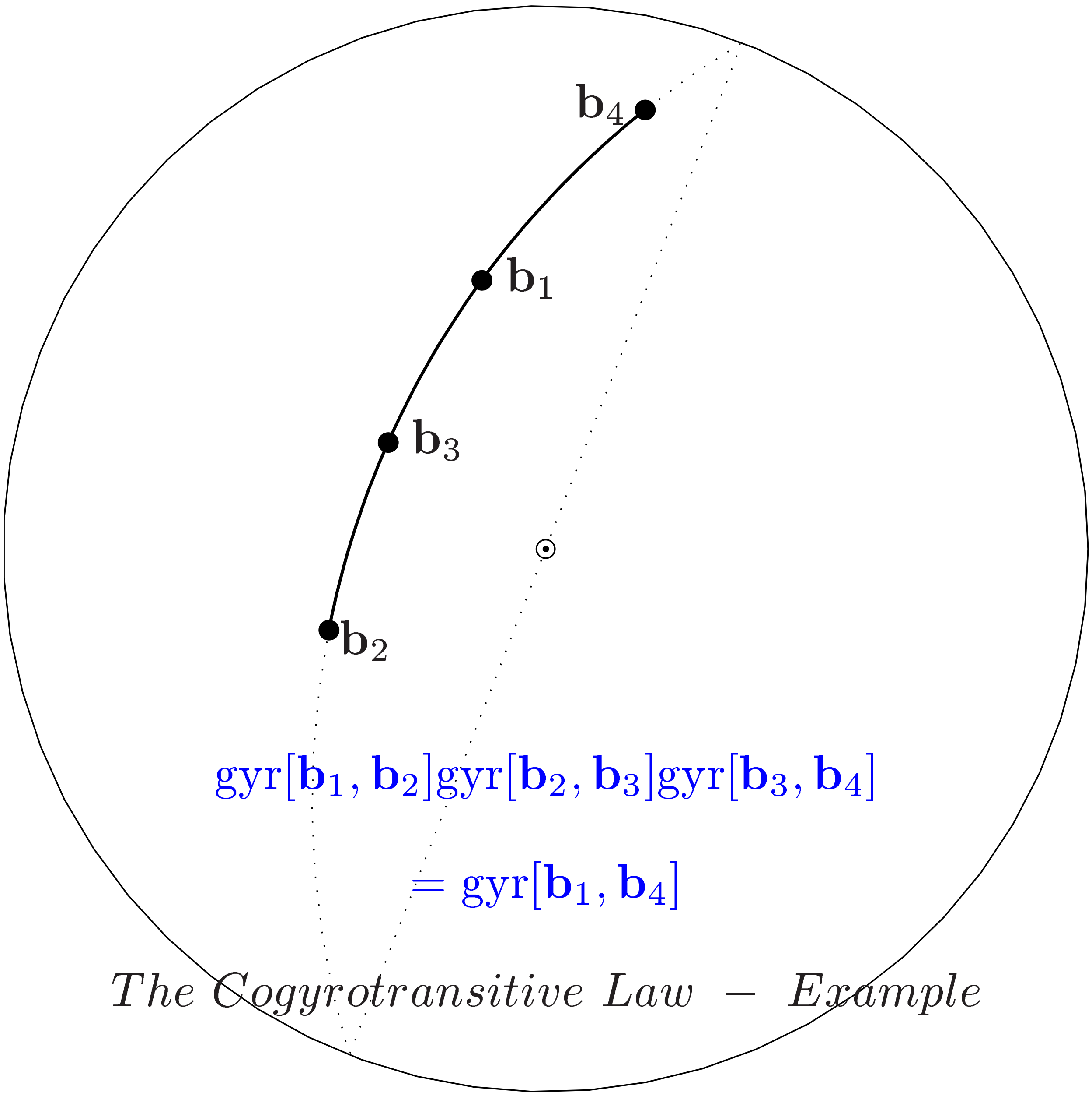

$\mathbf{b}_4$
$\mathbf{b}_1$
$\mathbf{b}_3$
$\mathbf{b}_2$
$\mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_2]\mathrm{gyr}[\mathbf{b}_2, \mathbf{b}_3]\mathrm{gyr}[\mathbf{b}_3, \mathbf{b}_4]$
$= \mathrm{gyr}[\mathbf{b}_1, \mathbf{b}_4]$
The Cogyrotransitive Law − Example

Figure 6:

The Hyperbolic Angle
and
The Hyperbolic Dual Angle

$$\boxed{\cos\alpha = \frac{\ominus\mathbf{a}\oplus\mathbf{b}}{\|\ominus\mathbf{a}\oplus\mathbf{b}\|}\cdot\frac{\ominus\mathbf{a}\oplus\mathbf{c}}{\|\ominus\mathbf{a}\oplus\mathbf{c}\|}} \tag{39}$$

Definition 7 (The Hyperbolic Dual Angle). *The measure of the hyperbolic dual angle* α *between two geometric dual gyrovectors* $\mathbf{b}\boxminus\mathbf{a}$ *and* $\mathbf{d}\boxminus\mathbf{c}$ *is given by the equation*

$$\boxed{\cos\alpha = \frac{\boxminus\mathbf{a}\boxplus\mathbf{b}}{\|\boxminus\mathbf{a}\boxplus\mathbf{b}\|}\cdot\frac{\boxminus\mathbf{c}\boxplus\mathbf{d}}{\|\boxminus\mathbf{c}\boxplus\mathbf{d}\|}} \tag{40}$$

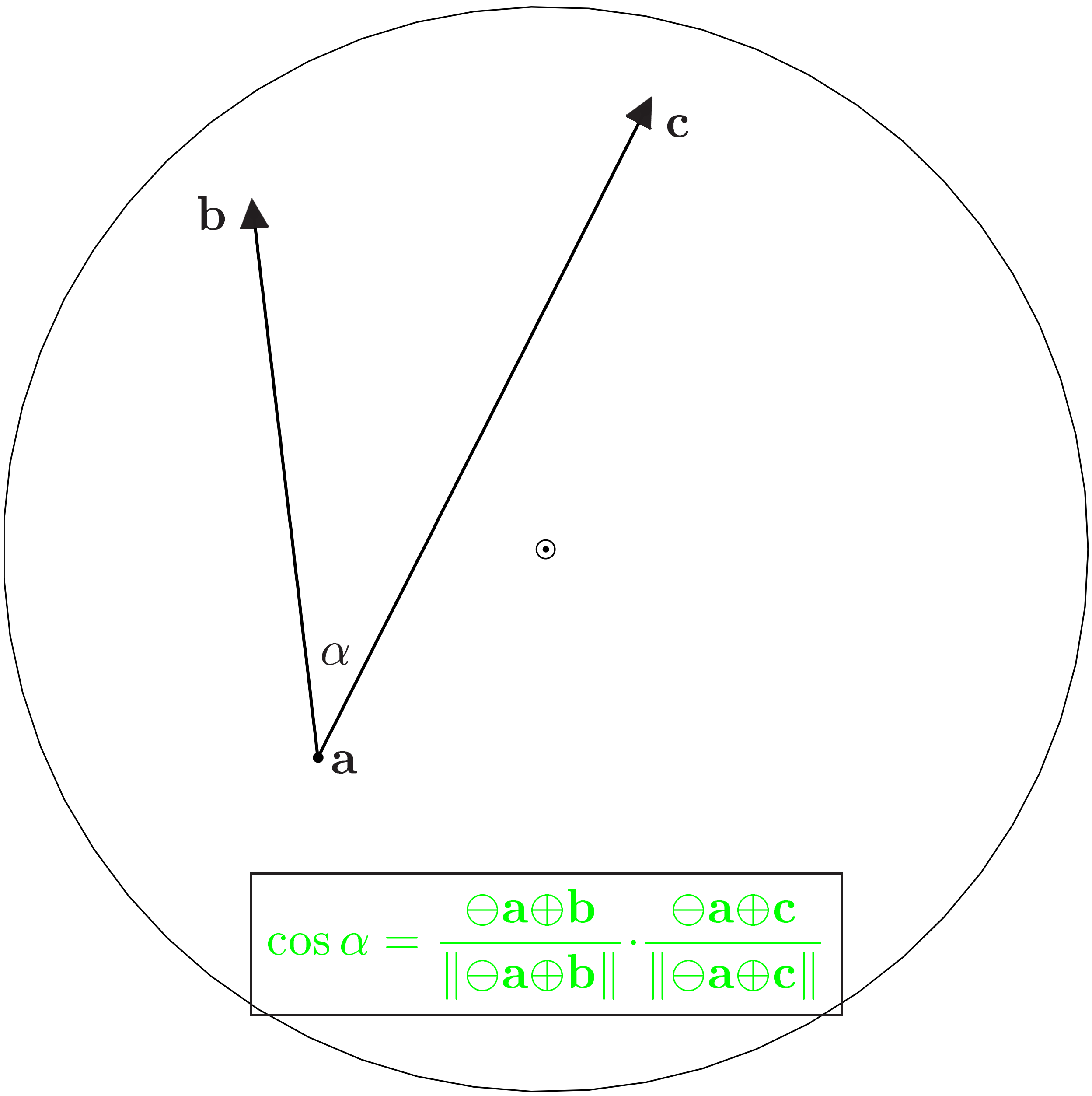


Figure 7: $\oplus = \oplus_{\rm E}$. The Hyperbolic Angle in the Einstein gyrovector plane = The Beltrami disc model of hyperbolic geometry.

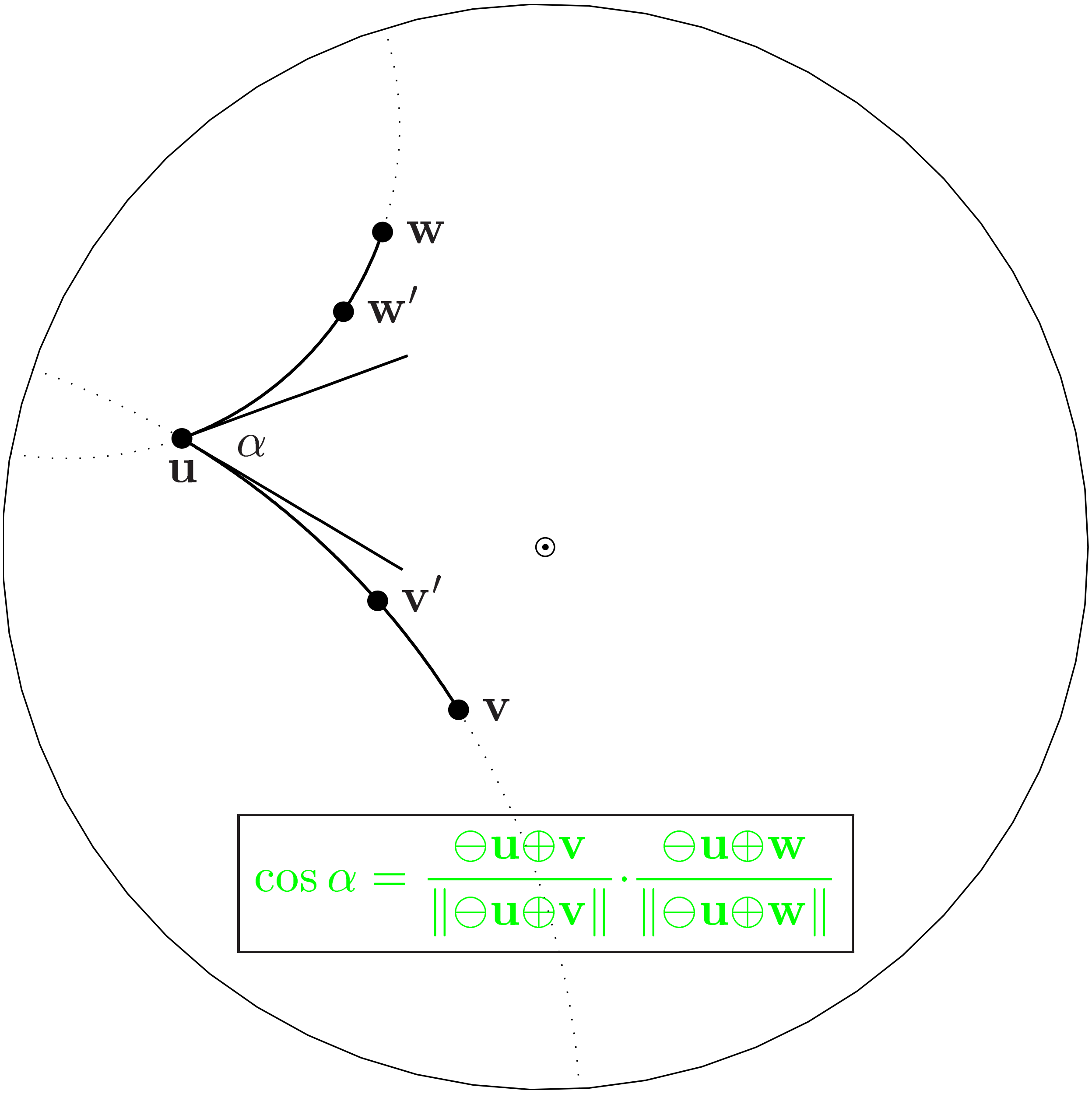


Figure 8: $\oplus = \oplus_{\text{M}}$. A Möbius angle α generated by the two intersecting Möbius geodesic rays. It is coincident with the hyperbolic angle of the Poincaré disc model of hyperbolic geometry.

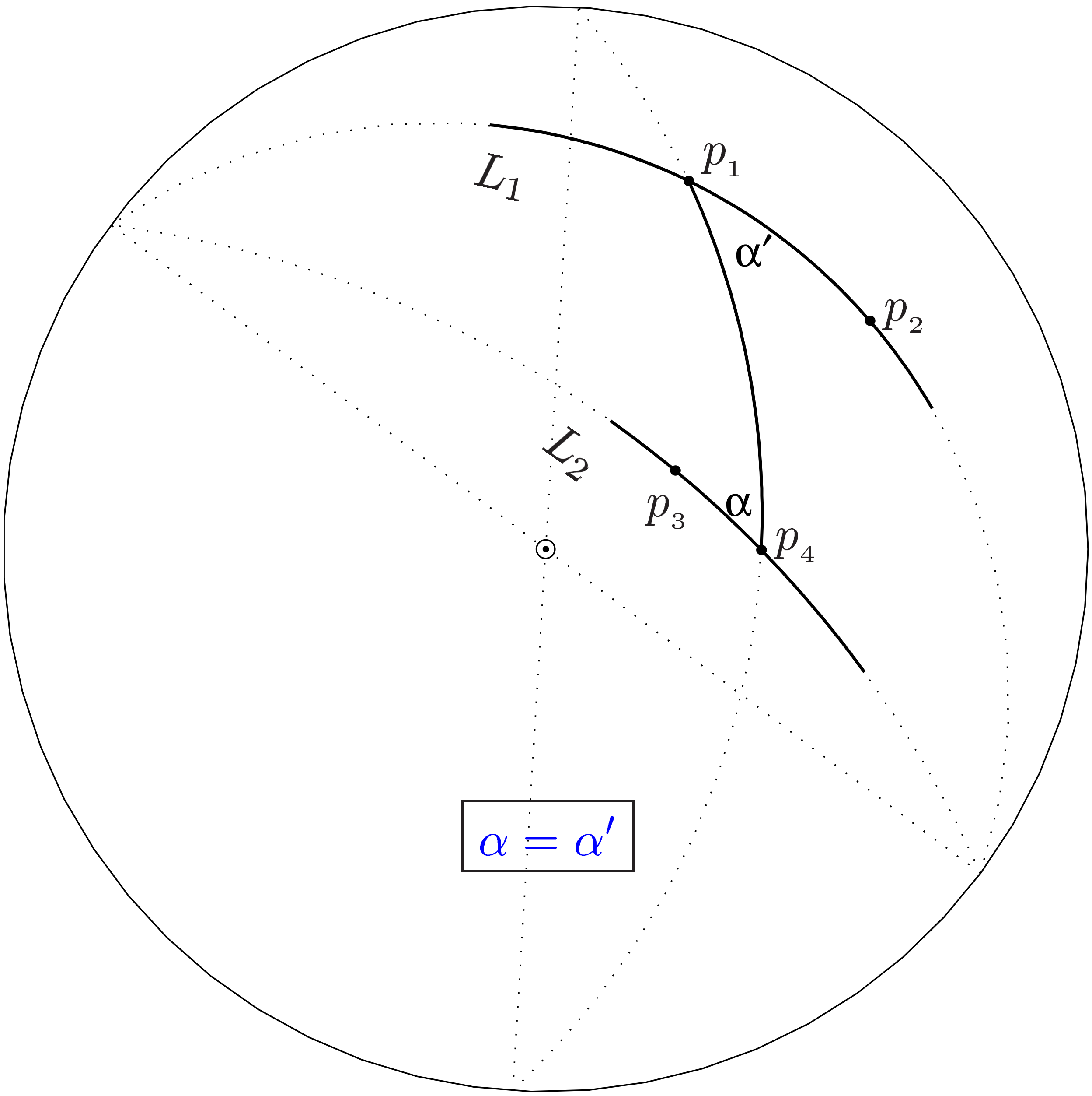


Figure 9: Hyperbolic Alternate Interior Coangles in a Möbius gyrovector space.

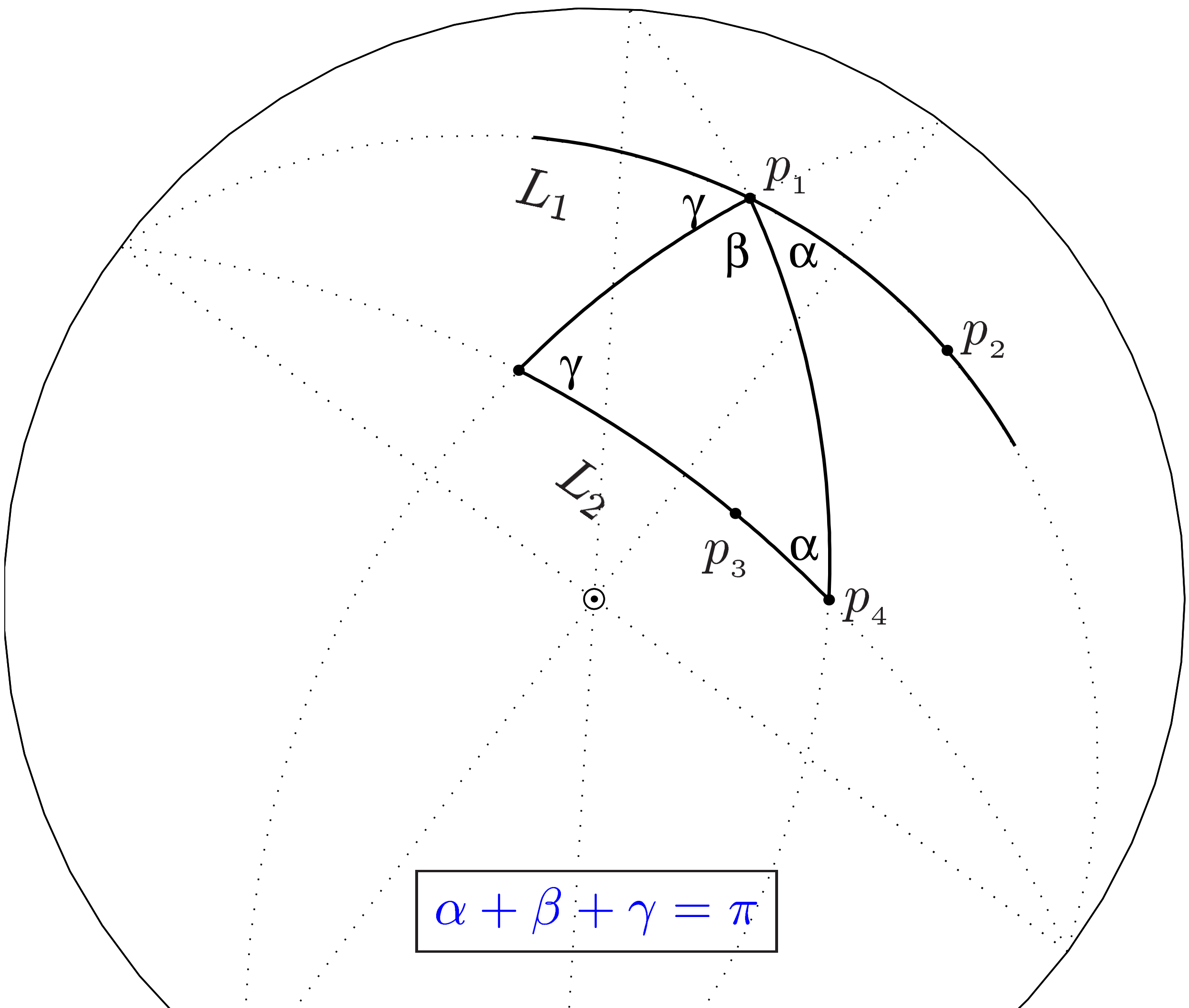


Figure 10: The Hyperbolic π Theorem in the Poincaré Disc Model: The cotriangle coangle sum is π.

Theorem 8 (The Hyperbolic π Theorem). *Let $\Delta\mathbf{abc}$ be the hyperbolic dual triangle in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ whose three vertices are the points $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{V}_c$. Then, its three hyperbolic dual angles α, β and γ, given by*

$$
\begin{aligned}
\cos\alpha &= \frac{\mathbf{b} \boxminus \mathbf{a}}{\|\mathbf{b} \boxminus \mathbf{a}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{a}}{\|\mathbf{c} \boxminus \mathbf{a}\|} \\
\cos\beta &= \frac{\mathbf{a} \boxminus \mathbf{b}}{\|\mathbf{a} \boxminus \mathbf{b}\|} \cdot \frac{\mathbf{c} \boxminus \mathbf{b}}{\|\mathbf{c} \boxminus \mathbf{b}\|} \\
\cos\gamma &= \frac{\mathbf{a} \boxminus \mathbf{c}}{\|\mathbf{a} \boxminus \mathbf{c}\|} \cdot \frac{\mathbf{b} \boxminus \mathbf{c}}{\|\mathbf{b} \boxminus \mathbf{c}\|}
\end{aligned}
\qquad (41)
$$

satisfy the identity

$$
\boxed{\alpha + \beta + \gamma = \pi} \qquad (42)
$$

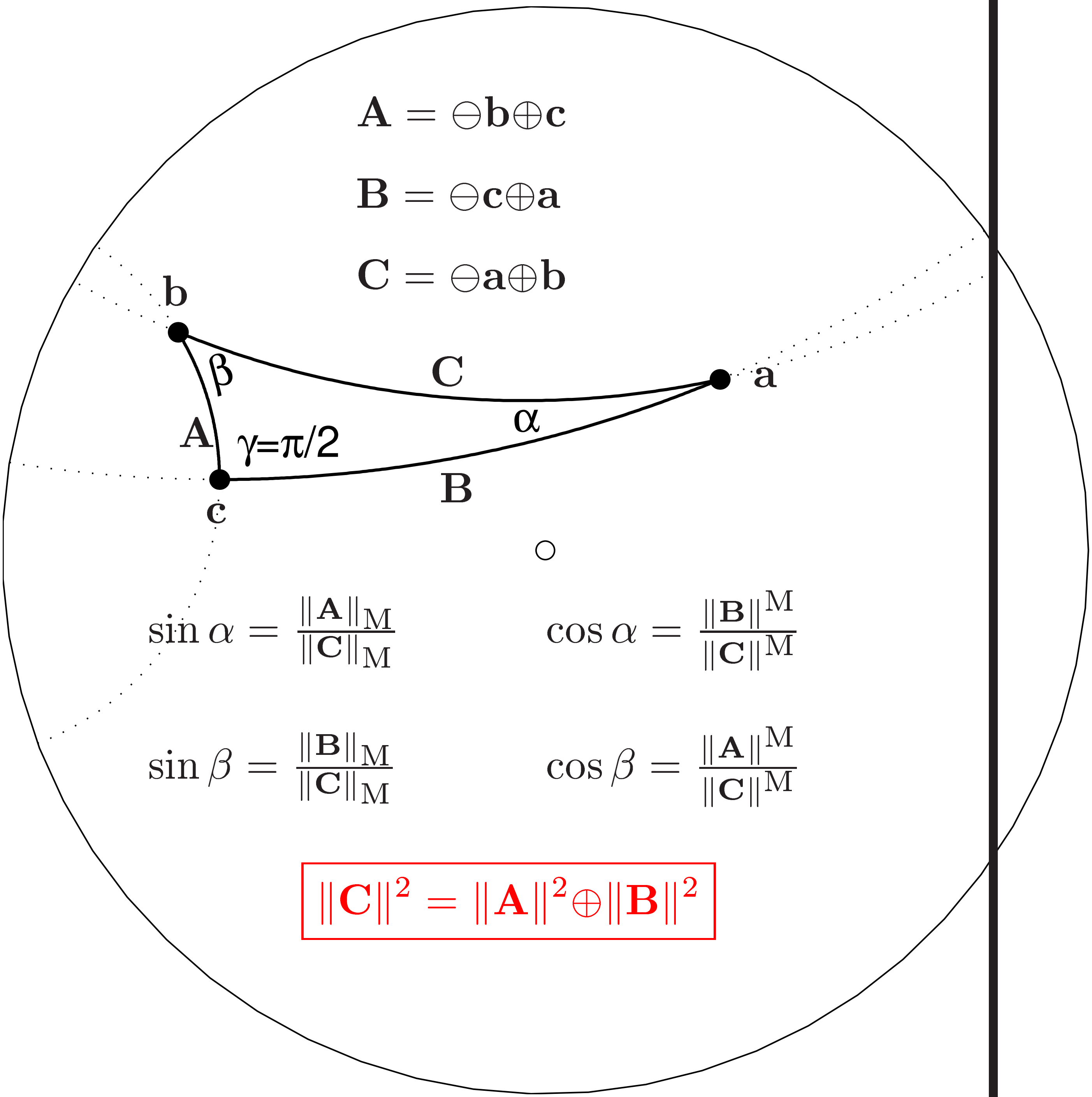


Figure 11: Hyperbolic trigonometry in Möbius gyrovector spaces.

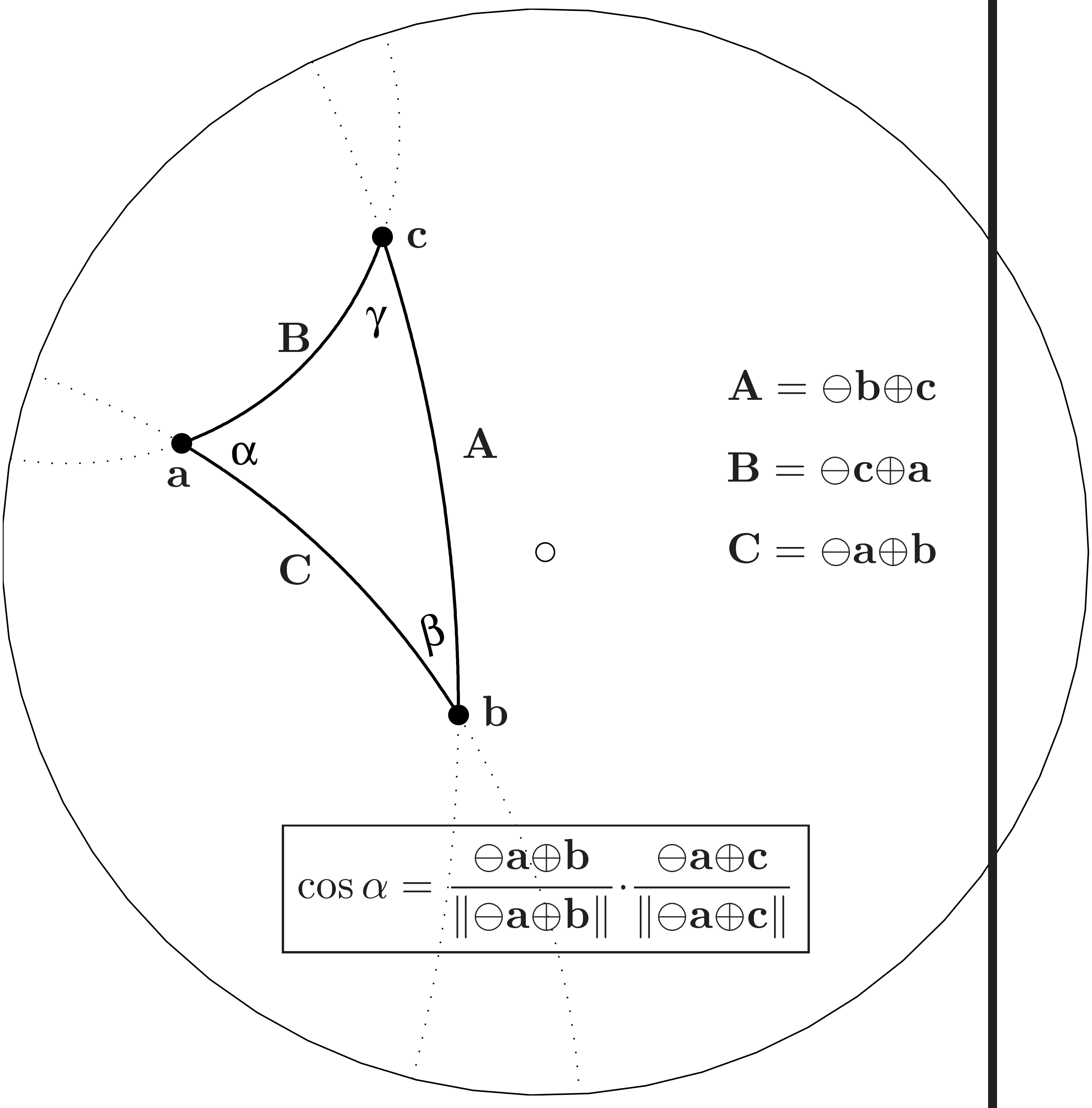


Figure 12: $\oplus = \oplus_{\mathrm{M}}$. A Möbius triangle $\boldsymbol{\Delta}\mathbf{abc}$ in the Möbius gyrovector plane $\mathbb{D} = (\mathbb{R}^2_c, \oplus_{\mathrm{M}}, \otimes)$.

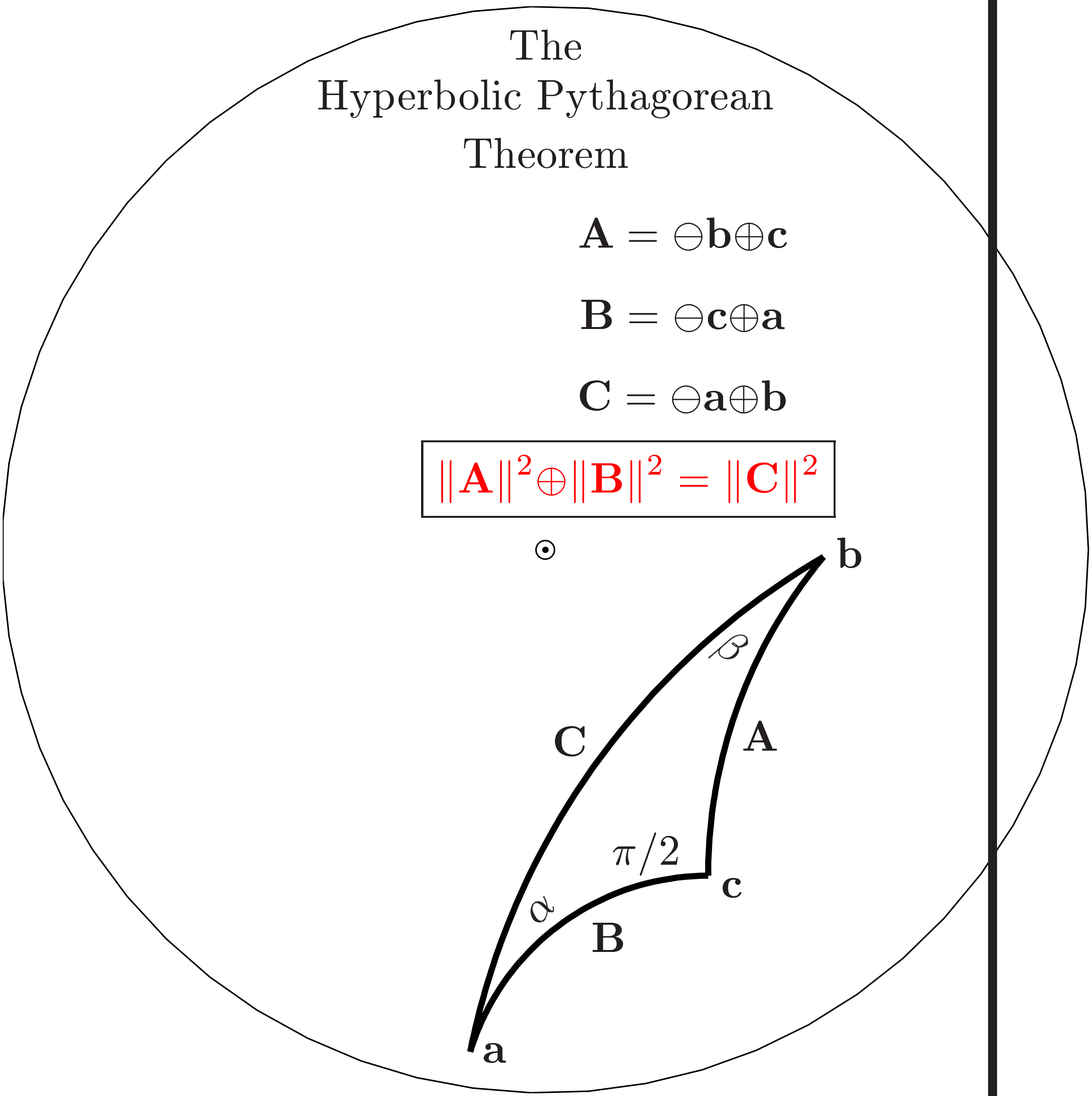


Figure 13: The hyperbolic Pythagorean theorem in Möbius gyrovector spaces.

$$\ominus\mathbf{a}_1\oplus\mathbf{b}_1 = \mathrm{gyr}[\mathbf{a}_1, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)$$

$$\ominus\mathbf{a}_2\oplus\mathbf{b}_2 = \mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_1](\ominus\mathbf{a}_1\oplus\mathbf{b}_1)$$

$$\ominus\mathbf{a}_2\oplus\mathbf{b}_2 = \mathrm{gyr}[\mathbf{a}_2, \ominus\mathbf{a}_0](\ominus\mathbf{a}_0\oplus\mathbf{b}_0)$$

$$\|\ominus\mathbf{a}_0\oplus\mathbf{b}_0\| = \|\ominus\mathbf{a}_1\oplus\mathbf{b}_1\| = \|\ominus\mathbf{a}_2\oplus\mathbf{b}_2\|$$

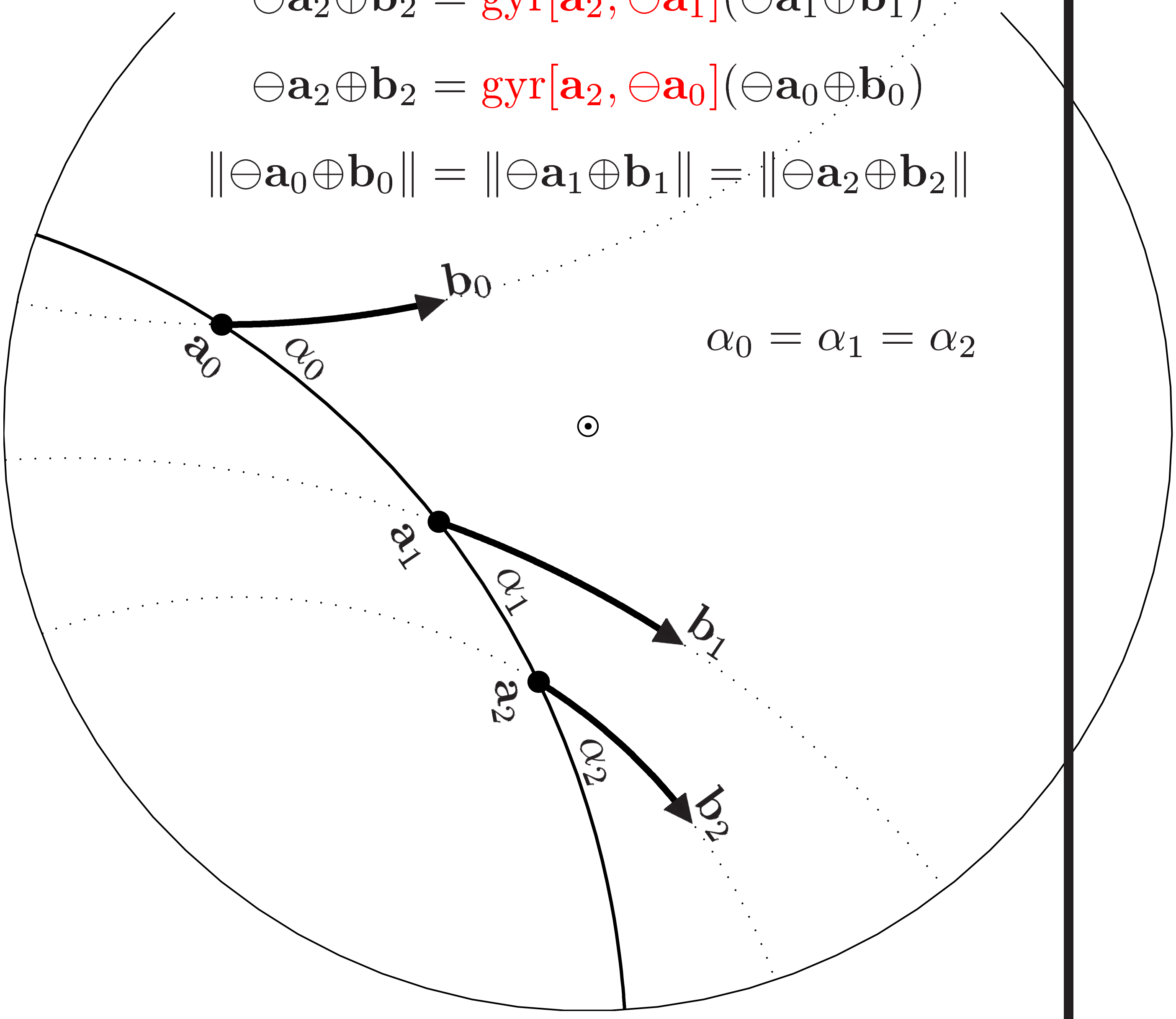


Figure 14: Hyperbolic parallel transport by (nonassociative) algebra rather than differential geometry.

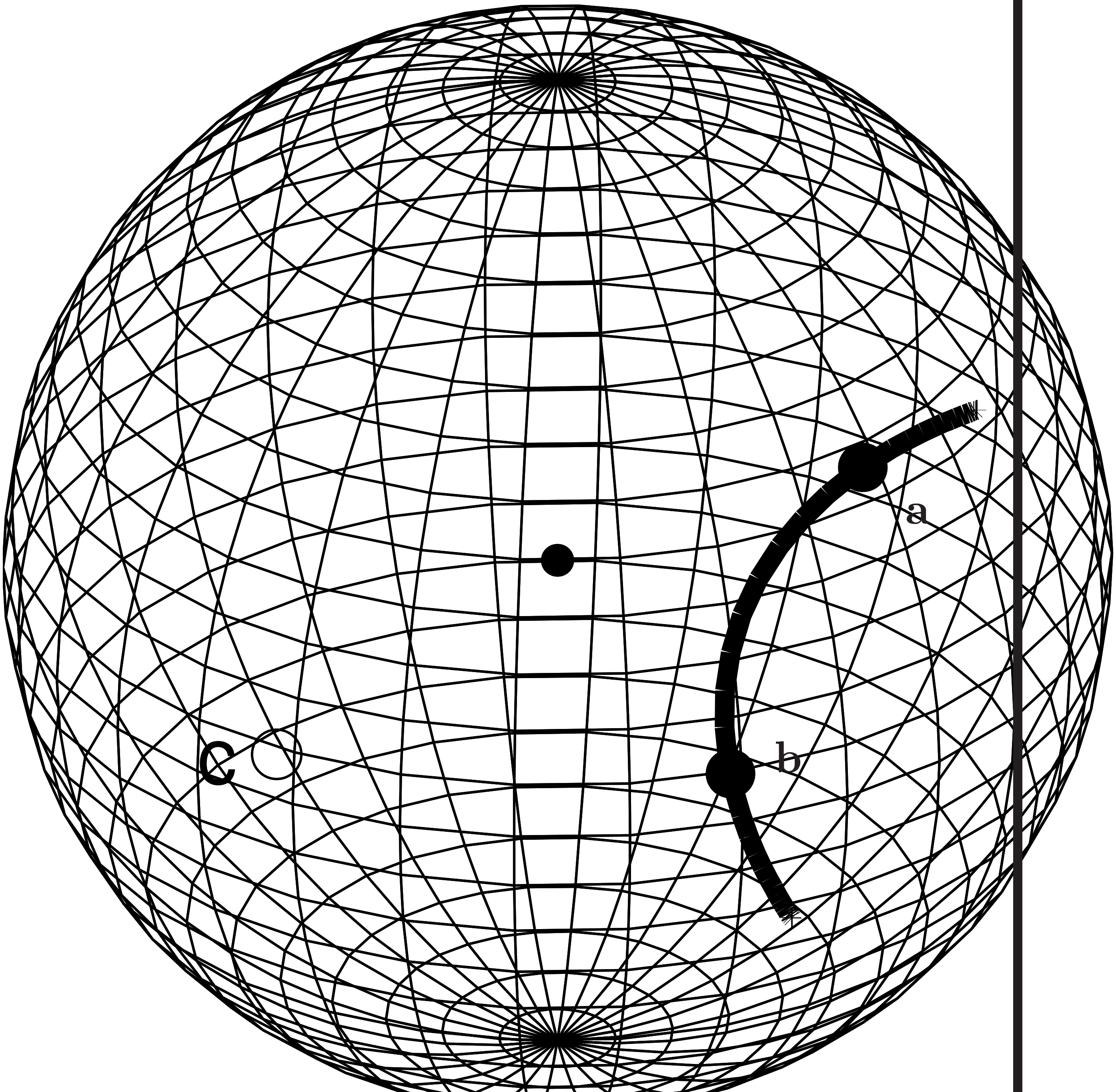


Figure 15: The 3-dimensional Möbius gyroline.

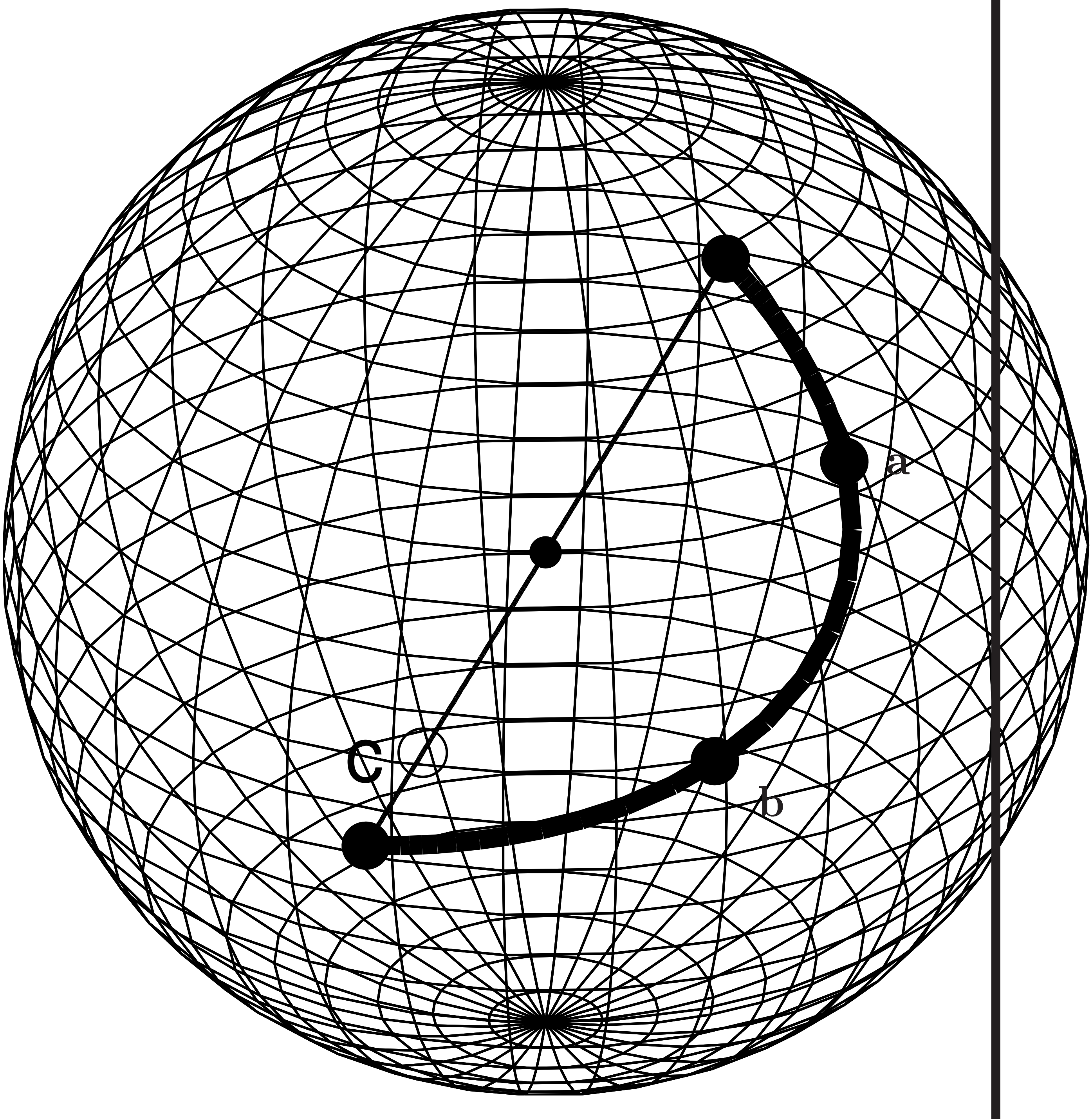
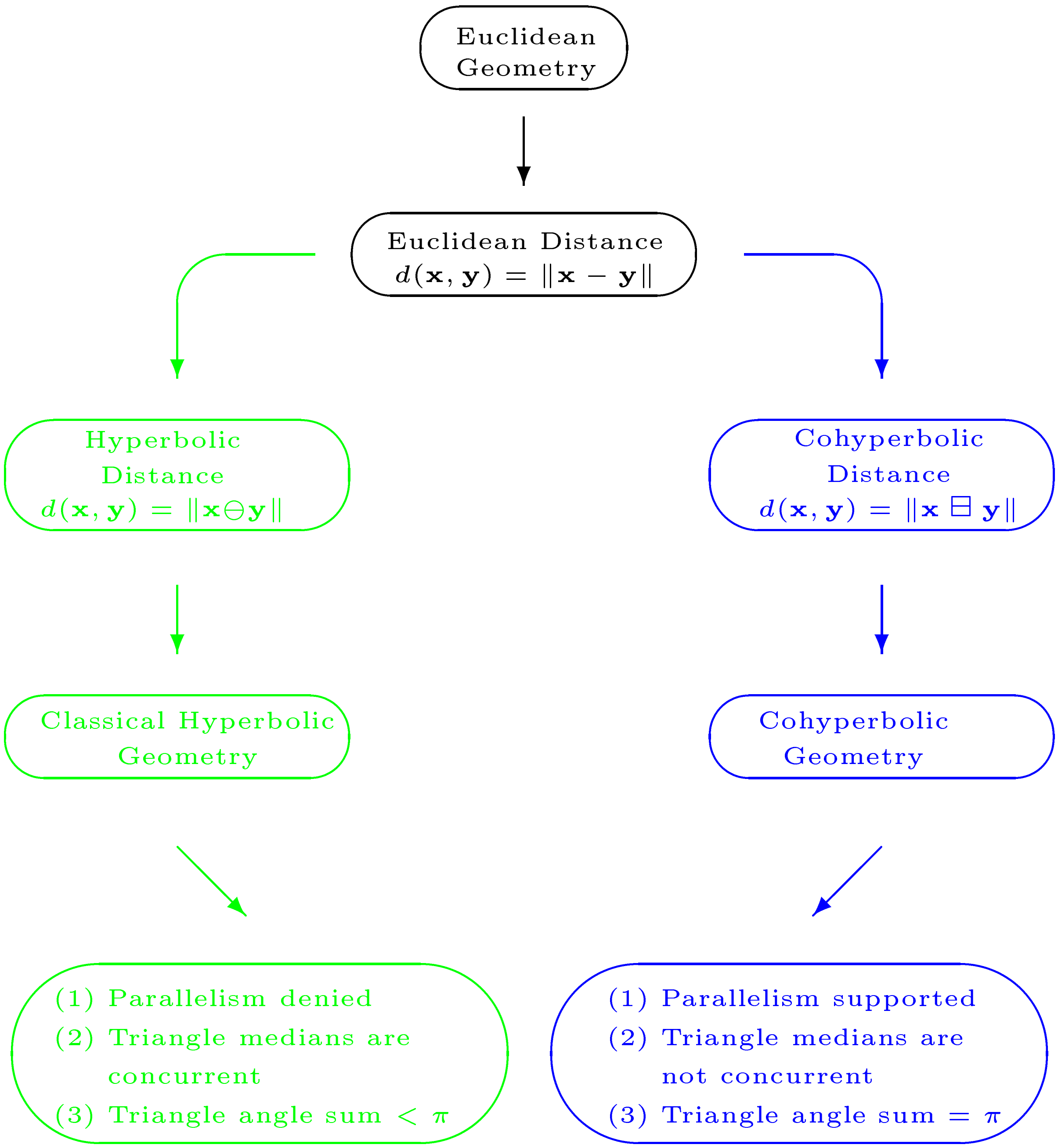


Figure 16: The 3-dimensional Möbius cogyroline.

The Hyperbolic Bifurcation



Gyrovector spaces
and
Riemannian Geometry

The Riemannian line element in a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is

$$\boxed{ds^2 = \|\ominus\mathbf{x}\oplus(\mathbf{x} + d\mathbf{x})\|^2} \tag{43}$$

Example I: When $\oplus = \oplus_{_\mathrm{M}}$, ds^2 turns out to be the Riemannian line element

$$\boxed{\begin{aligned} ds^2 &= \|\ominus_{_\mathrm{M}}\mathbf{x}\oplus_{_\mathrm{M}}(\mathbf{x} + d\mathbf{x})\|^2 \\ &= \frac{\sum_{i=1}^n dx_i^2}{\left(1 + \frac{1}{4}K\sum_{i=1}^n x_i^2\right)^2} \end{aligned}} \tag{44}$$

with curvature $K = -4/c^2$. For the special case when $n = 2$ this line element turns out to be the one of the $\boxed{\text{Poincaré}}$ ball model of hyperbolic geometry.

Example II: When $\oplus = \oplus_{\mathrm{E}}$, ds^2 turns out to be the Riemannian line element

$$\boxed{\begin{aligned} & ds^2 = \|\ominus_{\mathrm{E}}\mathbf{x}\oplus_{\mathrm{E}}(\mathbf{x}+d\mathbf{x})\|^2 = \\ & c^2\,\frac{\sum_{i=1}^{n}\{c^2-(r^2-x_i^2)\}dx_i^2 + 2\sum_{\substack{i,j=1\\ i<j}}^{n} x_i x_j dx_i dx_j}{(c^2-r^2)^2} \end{aligned}} \tag{45}$$

where $r^2 = \sum_{i=1}^{n} x_i^2$. For the special case when $n = 2$ this line element turns out to be the one of the $\boxed{\text{Beltrami}}$ (or Klein) ball model of hyperbolic geometry.

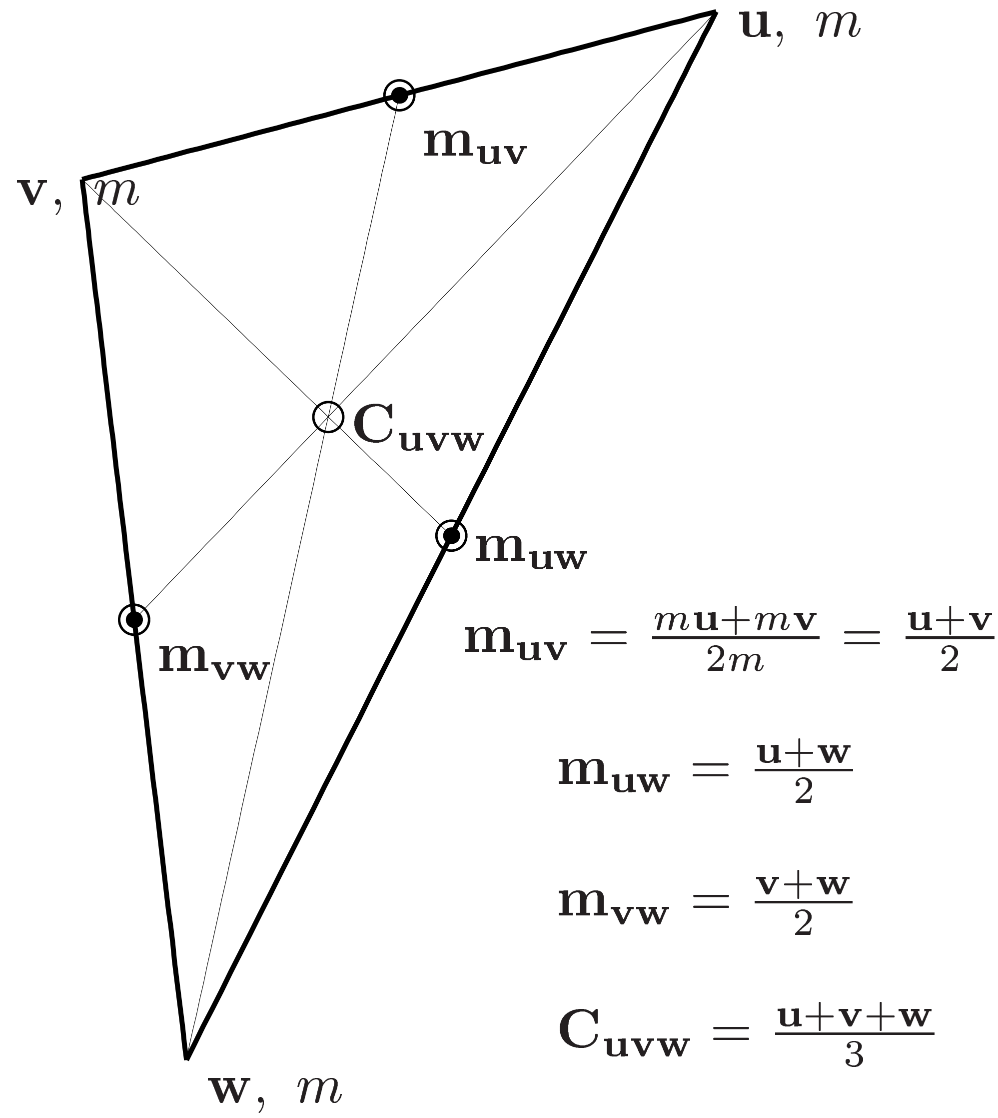


Figure 17: The Euclidean Centroid.

A massive object with mass m is located at each of the three vertices $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ of triangle $\Delta\mathbf{uvw}$ in the Euclidean 3-space $\mathbb{R}^3$ of Newtonian velocities. The midoints of the sides of triangle $\Delta\mathbf{uvw}$ are $\mathbf{m_{uv}}$, $\mathbf{m_{uw}}$ and $\mathbf{m_{vw}}$. The centroid $\mathbf{C_{uvw}}$ of triangle $\Delta\mathbf{uvw}$ is equal to the velocity of the center of mass of the three massive objects with equal masse s m that coincided at time $t = 0$, and are moving with Nwtonian velocities $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ relative to some inertial rest frame. A straightforward extension to the relativistic regime is shown in Fig. 18.

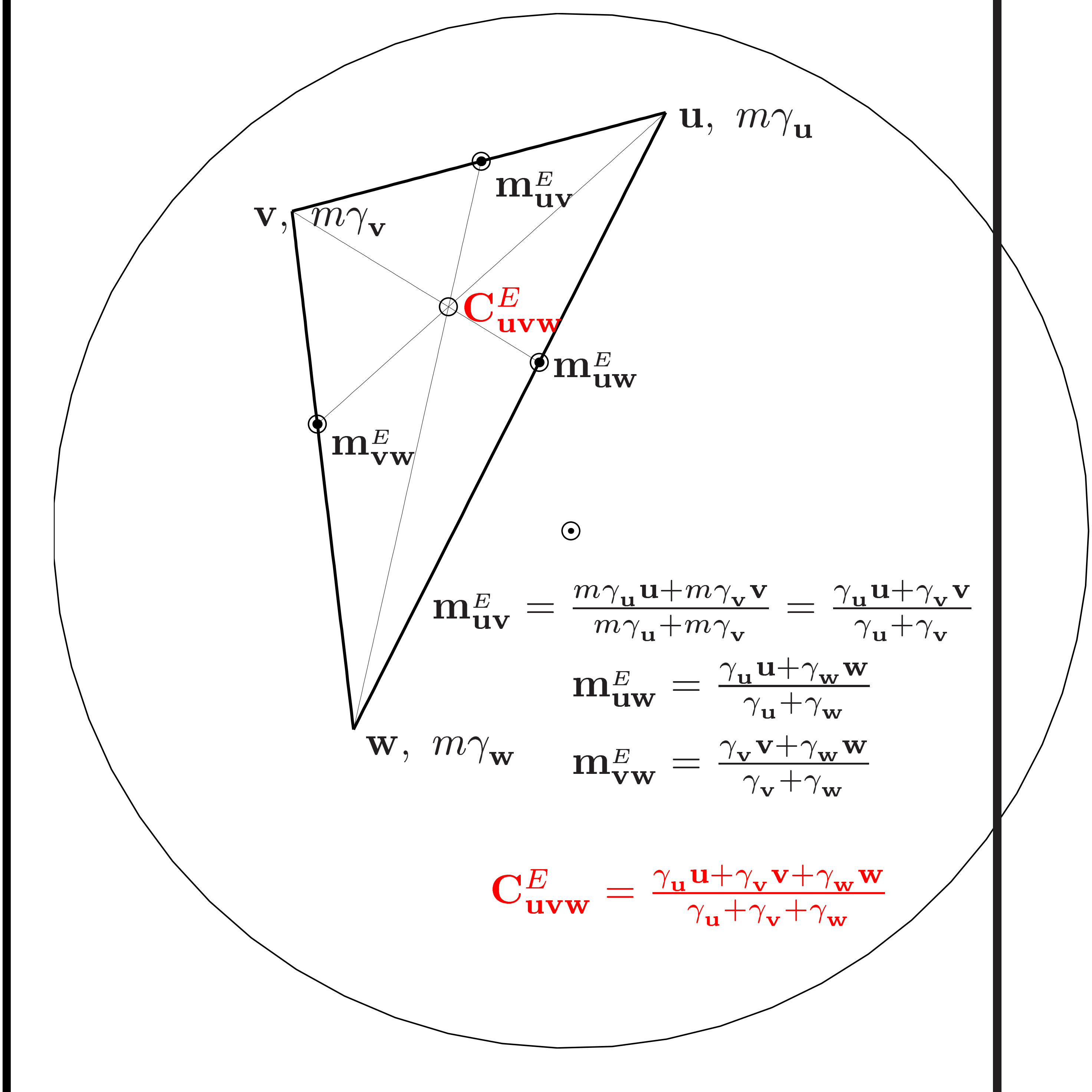


Figure 18: The Einstein Centroid

A massive object with relativistic rest mass m is located at each of the three vertices $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ of triangle $\Delta\mathbf{uvw}$ in the hyperbolic 3-space $\mathbb{R}_c^3$ of Einsteinian velocities, that is, in the Beltrami 3-dimensional c-ball model of hyperbolic geometry. The relativistic masses at the vertices $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ corresponding to their equal rest masses m are, according to the special theory of relativity, $m\gamma_{\mathbf{u}}$, $m\gamma_{\mathbf{v}}$ and $m\gamma_{\mathbf{w}}$. The midoints of the sides of triangle $\Delta\mathbf{uvw}$ are $\mathbf{m}_{\mathbf{uv}}^{E}$, $\mathbf{m}_{\mathbf{uw}}^{E}$ and $\mathbf{m}_{\mathbf{vw}}^{E}$. Interestingly, the hyperbolic centroid $\mathbf{C}_{\mathbf{uvw}}^{E}$ of triangle $\Delta\mathbf{uvw}$ is equal to the velocity of the center of mass of the three massive objects with equal rest masses m that coincided at time $t = 0$, and are moving with Einsteinian velocities $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ relative to the inertial rest frame $\Sigma_{\mathbf{0}}$.

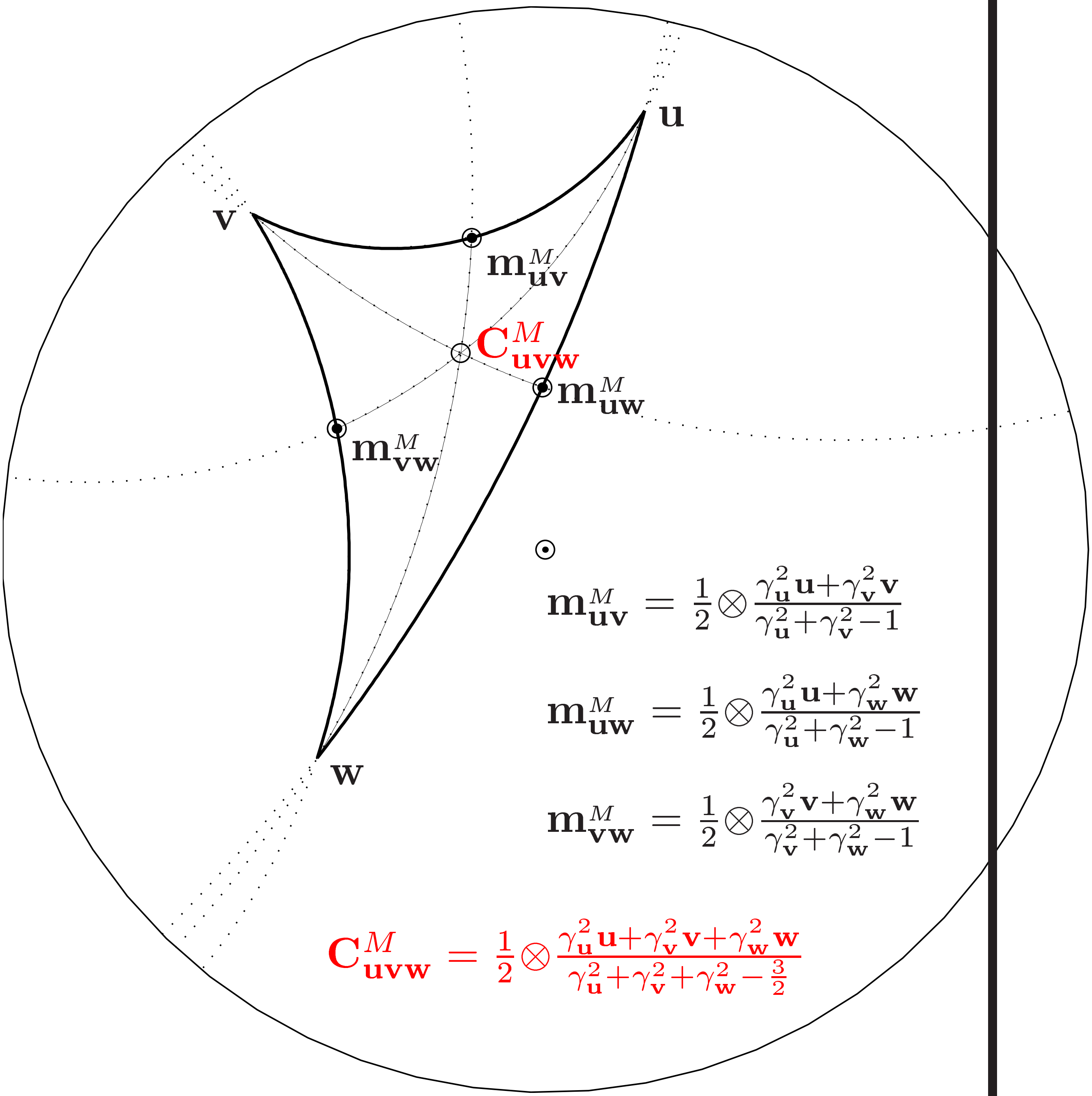


Figure 19: The Möbius Centroid

A triangle $\Delta\mathbf{uvw}$ in the Poincaré disc model $\mathbb{R}_c^2$ of hyperbolic geometry is shown with the midpoints $\mathbf{m}_{\mathbf{uv}}^{M}$, $\mathbf{m}_{\mathbf{uw}}^{M}$ and $\mathbf{m}_{\mathbf{vw}}^{M}$ of its sides, and its centroid $\mathbf{C}_{\mathbf{uvw}}^{M}$.

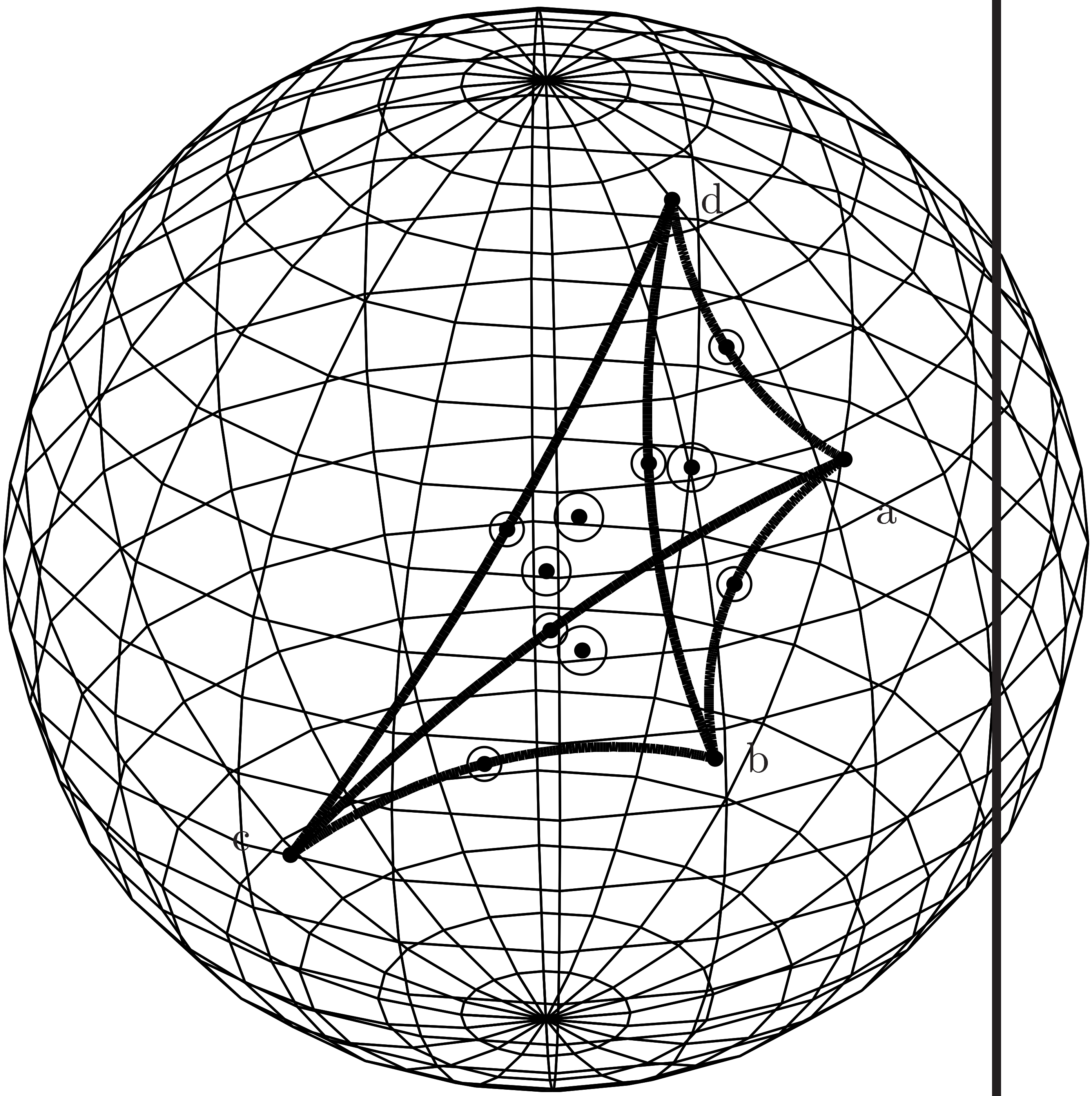


Figure 20: The hyperbolic tetrahedron $\Delta\mathbf{abcd}$, called a gyrotetrahedron.

The hyperbolic tetrahedron $\mathbf{\Delta abcd}$, called a gyrotetrahedron, is shown in the Möbius gyrovector space $\mathbb{R}^3_c = (\mathbb{R}^3_c, \oplus_{\text{M}}, \otimes_{\text{M}})$ underlying the Poincarè ball model of hyperbolic geometry, with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d} \in \mathbb{R}^3_c$. The gyrotetrahedron $\mathbf{\Delta abcd}$ is shown inside the c-ball $\mathbb{R}^3_c$ of the Euclidean 3-space $\mathbb{R}^3$ where it lives. The faces of the gyrotetrahedron are gyrotriangles.

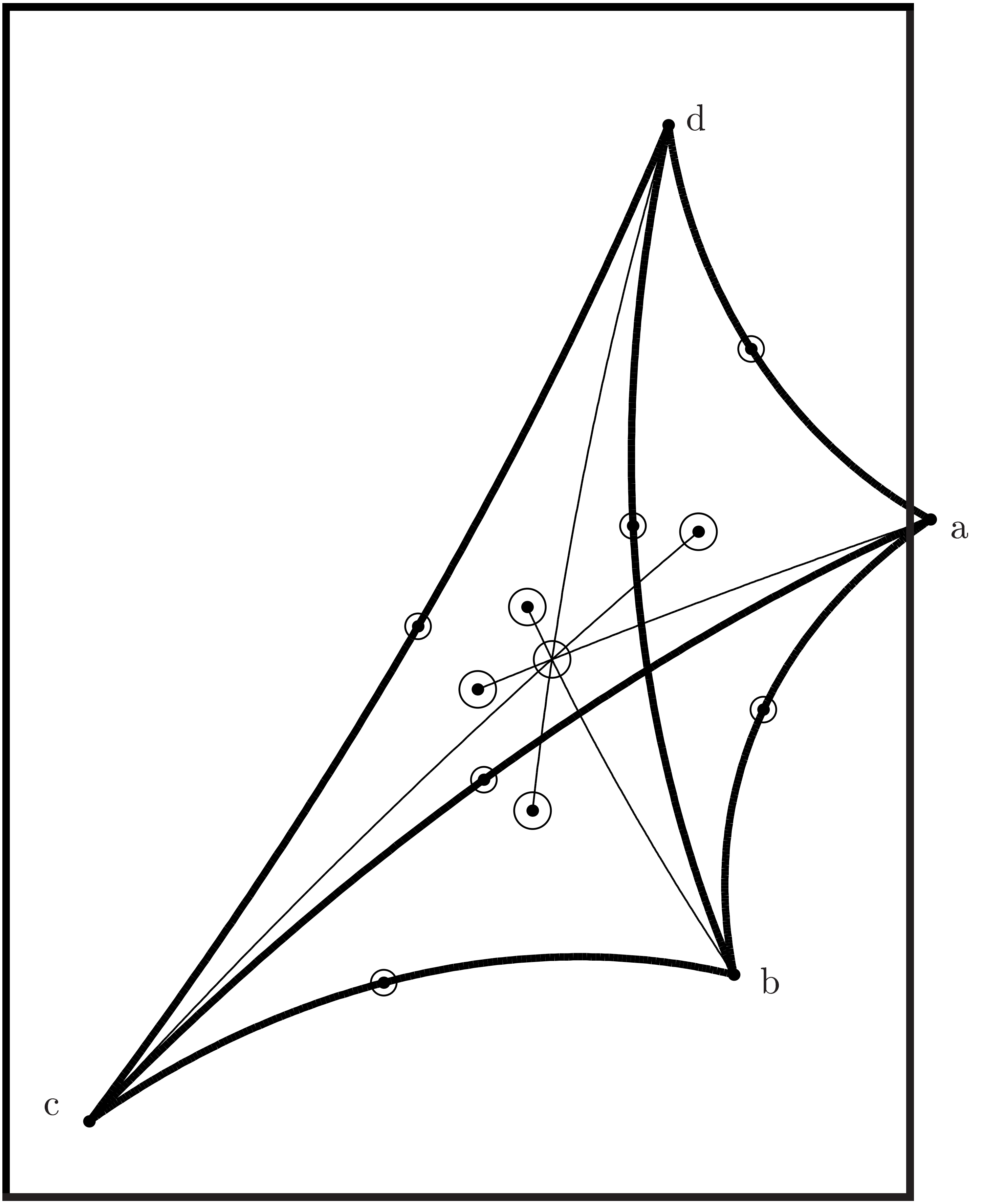


Figure 21:

The gyroline joining a vertex of a gyrotetrahedron and the hyperbolic centroid, gyrocentroid, of the opposite face is called a gyrotetrahedron median. The four medians of the gyrotetrahedron $\Delta\mathbf{abcd}$, determined by the four indicated centroids are concurrent. They are shown with their common point, which is the gyrotetrahedron centroid $C^M_{\mathbf{abcd}}$.

The Gyro-Structure of The Density Matrix For Mixed State Qubits

A *qubit* is a two state quantum system completely described by the qubit density matrix $\rho(\mathbf{v})$,

$$\rho_{\mathbf{v}} = \tfrac{1}{2}\begin{pmatrix} 1+v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix} \qquad (46)$$

parametrized by the vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{B}^3$, where $\mathbb{B}^3$ is the open unit ball of the Euclidean 3-space $\mathbb{R}^3$. The parameter $\mathbf{v}$ is known as the Bloch vector and, accordingly, the ball $\mathbb{B}^3$ is called the Bloch ball.

The set of all *qubit mixed state density matrices* is a gyrovector space isomorphic copy of the 3-dimensional Möbius gyrovector space. One of the resulting identities is, for instance,

$$\boxed{\rho_{\mathbf{v}}^{r} = tr(\rho_{\mathbf{v}}^{r})\rho_{r\otimes\mathbf{v}}} \qquad (47)$$

for all $\mathbf{v}\in\mathbb{B}^3$ and $r\in\mathbb{R}$.

$$\rho(\mathbf{v}) = \tfrac{1}{2}\begin{pmatrix} 1+v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix} \tag{48}$$

The Bloch vector $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{B}^3$ is, in fact, a gyrovector in the Möbius ball gyrovector space.

Another elegant identiy, demonstrating that the home of the Bloch vector of Quantum Computation is the 3-dimensional Möbius gyrovector space is the following.

$$\boxed{\rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}} = tr(\rho_{\mathbf{u}}\rho_{\mathbf{v}}^2\rho_{\mathbf{u}})\rho_{2\otimes(\mathbf{u}\oplus\mathbf{v})}} \tag{49}$$

where $\oplus$ and $\otimes$ are Möbius addition and scalar multiplication.

This identity expresses a matrix product of several qubit density matrices as a single density matrix with a positive coefficient.

Gyrogroup-theoretic techniques capture analogies between the Galilei and the Lorentz transformation groups as well:

$$\boxed{L(\mathbf{v},V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} \gamma_{\mathbf{v}}(t+\frac{1}{c^2}\mathbf{v}\cdot V\mathbf{x}) \\ \gamma_{\mathbf{v}}\mathbf{v}t + V\mathbf{x} + \frac{1}{c^2}\frac{\gamma_{\mathbf{v}}^2}{1+\gamma_{\mathbf{v}}}(\mathbf{v}\cdot V\mathbf{x})\mathbf{v} \end{pmatrix}} \tag{50}$$

$\mathbf{v}\in\mathbb{R}_c^n$, $V\in SO(n)$, $\mathbf{x}\in\mathbb{R}^n$, $t\in\mathbb{R}^+$.

In the limit of large vacuum speed of light, $c\to\infty$, the Lorentz transformation $L(\mathbf{v},V)$ in (50) reduces to its Galilean counterpart $G(\mathbf{v},V)$,

$$\boxed{G(\mathbf{v},V)\begin{pmatrix} t \\ \mathbf{x} \end{pmatrix} = \begin{pmatrix} t \\ \mathbf{v}t+V\mathbf{x} \end{pmatrix}} \tag{51}$$

$\mathbf{v}\in\mathbb{R}^n$, $V\in SO(n)$, $\mathbf{x}\in\mathbb{R}^n$, $t\in\mathbb{R}$.

The Galilei Transformation Composition Law Is Given By Parameter Composition:

$$\boxed{G(\mathbf{u}, U)G(\mathbf{v}, V) = G(\mathbf{u} + U\mathbf{v}, UV)} \tag{52}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $U, V \in SO(n)$. The Lorentz

Transformation Composition Law Is Given By Parameter Composition:

$$\boxed{L(\mathbf{u}, U)L(\mathbf{v}, V) = L(\mathbf{u} \oplus U\mathbf{v}, \mathrm{gyr}[\mathbf{u}, U\mathbf{v}]UV)} \tag{53}$$

where $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, and $U, V \in SO(n)$.

The set of all Galilei transformations that link any two given "equitime" spacetime events is well known:

$$\boxed{G(\mathbf{v} - R\mathbf{u}, R) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix}} \tag{54}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, and $t, t' \in \mathbb{R}$, where $R \in SO(n)$ is a free rotation parameter.

In full analogy, the set of all Lorentz transformations that link any two given "equinorm" spacetime events is, surprisingly:

$$\boxed{L(\mathbf{v} \boxminus R\mathbf{u}, R) \begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix}} \tag{55}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, and $t, t' \in \mathbb{R}^+$, where $R \in SO(n)$ is a free rotation parameter.

The gyrogroup dual operation $\boxplus$ thus plays centrally in the ability of gyrogroup-theoretic techniques to capture analogies.

	Composition Law	binary operation
Galilei	$\boxed{G(\mathbf{u},U)G(\mathbf{v},V) = G(\mathbf{u}+U\mathbf{v}, UV)}$ $\mathbf{u},\mathbf{v} \in \mathbb{R}^n, \quad U,V \in SO(n)$ Structure: Semidirect Product Group	$+$
Lorentz	$\boxed{L(\mathbf{u},U)L(\mathbf{v},V) = L(\mathbf{u}\oplus U\mathbf{v}, \mathrm{gyr}[\mathbf{u},U\mathbf{v}]UV)}$ $\mathbf{u},\mathbf{v} \in \mathbb{R}^n_c, \quad U,V \in SO(n)$ Structure: Gyrosemidirect Product Group	$\oplus$
	Link	
Galilei	$\boxed{G(\mathbf{v}-R\mathbf{u}, R)\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix}}$ $t' = t$ $R \in SO(n)$ *free orientation parameter*	$-$
Lorentz	$\boxed{L(\mathbf{v}\boxminus R\mathbf{u}, R)\begin{pmatrix} t \\ \mathbf{u}t \end{pmatrix} = \begin{pmatrix} t' \\ \mathbf{v}t' \end{pmatrix}}$ $t'\sqrt{1-\Vert\mathbf{v}\Vert^2/c^2} = t\sqrt{1-\Vert\mathbf{u}\Vert^2/c^2}$ $R \in SO(n)$ *free orientation parameter*	$\boxminus$

Table 1: Galilei and Lorentz transformation analogies in their composition law and links. Remarkably, in order to capture analogies the two dual Einstein additions, $\oplus$ and $\boxplus$, are employed.

Those who want to know more about gyrogroups and gyrovector spaces will find it rewarding to read the new book

Beyond
The Einstein Addition Law
And Its
Gyroscopic Thomas Precession
The Theory Of
Gyrogroups And Gyrovector
Spaces

by A.A. Ungar (Kluwer Acad. 2001)

See URL:
www.wkap.nl/book.htm/0-7923-6909-2