# Homework # 3 Solutions

February 11, 2010

**Solution** (2.3.5)**.** Noting that

$$\begin{aligned}
&\lim_{x\to 8}(1+\sqrt[3]{x}) \\
=&\lim_{x\to 8} 1 + \lim_{x\to 8}\sqrt[3]{x}) && \text{by Equation (2.3.1)} \\
=&1+\sqrt[3]{8} && \text{by Equations (2.3.7) and (2.3.10)} \\
=&3
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{x\to 8}(2-6x^2+x^3) \\
=&\lim_{x\to 8} 2 + \lim_{x\to 8} -6x^2 + \lim_{x\to 8} x^3 && \text{by Equation (2.3.1)} \\
=&\lim_{x\to 8} 2 - 6\lim_{x\to 8} x^2 + \lim_{x\to 8} x^3 && \text{by Equation (2.3.3)} \\
=&2-6(8^2)+8^3 && \text{by Equations (2.3.7) and (2.3.10)} \\
=&130
\end{aligned}$$

(this could also have been accomplished using the "Direct Substitution Property") we find that

$$\begin{aligned}
&\lim_{x\to 8}(1+\sqrt[3]{x})(2-6x^2+x^3) \\
=&\left[\lim_{x\to 8}(1+\sqrt[3]{x})\right]\left[\lim_{x\to 8}(2-6x^2+x^3)\right] && \text{by Equation (2.3.4)} \\
=&[3][130]=390.
\end{aligned}$$

**Solution** (2.3.7)**.** Noting that

$$\begin{aligned}
&\lim_{x\to 1}(1+3x) \\
=&\lim_{x\to 1} 1 + \lim_{x\to 1} 3x && \text{by Equation (2.3.1)} \\
=&\lim_{x\to 1} 1 + 3\lim_{x\to 1} x && \text{by Equation (2.3.3)} \\
=&1+3(1) && \text{by Equations (2.3.7) and (2.3.8)} \\
=&4
\end{aligned}$$

and

$$\begin{aligned}
&\lim_{x\to 1}(1+4x^2+3x^4) \\
&=\lim_{x\to 1} 1+\lim_{x\to 1} 4x^2+\lim_{x\to 1} 3x^4 && \text{by Equation (2.3.1)} \\
&=\lim_{x\to 1} 2+4\lim_{x\to 1} x^2+3\lim_{x\to 1} x^4 && \text{by Equation (2.3.3)} \\
&=1+4(1^2)+3(1^4) && \text{by Equations (2.3.7) and (2.3.10)} \\
&=8
\end{aligned}$$

(both of these could also have been accomplished using the "Direct Substitution Property") we find that

$$\begin{aligned}
&\lim_{x\to 1}\frac{1+3x}{1+4x^2+3x^4} \\
&=\frac{\lim_{x\to 1}(1+3x)}{\lim_{x\to 1}(1+4x^2+3x^4)} && \text{by Equation (2.3.5)} \\
&=\frac{4}{8}=\frac{1}{2}.
\end{aligned}$$

**Solution** (2.3.15)**.**

$$\lim_{t\to -3}\frac{t^2-9}{2t^2+7t+3}=\lim_{t\to -3}\frac{(t-3)(t+3)}{(t+3)(2t+1)}=\lim_{t\to -3}\frac{(t-3)}{(2t+1)}=\frac{-6}{-5}=\frac{6}{5}.$$

**Solution** (2.3.19)**.**

$$\lim_{x\to -2}\frac{x+2}{x^3+8}=\lim_{x\to -2}\frac{x+2}{(x+2)(x^2-2x+4)}=\lim_{x\to -2}\frac{1}{(x^2-2x+4)}=\frac{1}{12}.$$

**Solution** (2.3.23)**.**

$$\begin{aligned}
\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}&=\lim_{x\to 7}\frac{\sqrt{x+2}-3}{x-7}\cdot\frac{\sqrt{x+2}+3}{\sqrt{x+2}+3}=\lim_{x\to 7}\frac{(x+2)-9}{(x-7)(\sqrt{x+2}+3)} \\
&=\lim_{x\to 7}\frac{x-7}{(x-7)(\sqrt{x+2}+3)}=\lim_{x\to 7}\frac{1}{\sqrt{x+2}+3}=\frac{1}{\sqrt{9}+3}=\frac{1}{6}.
\end{aligned}$$

**Solution** (2.3.25)**.**

$$\lim_{x\to -4}\frac{\frac{1}{4}+\frac{1}{x}}{4+x}=\lim_{x\to -4}\frac{\left(\frac{x+4}{4x}\right)}{4+x}=\lim_{x\to -4}\left[\left(\frac{x+4}{4x}\right)\cdot\frac{1}{4+x}\right]=\lim_{x\to -4}\frac{1}{4x}=-\frac{1}{16}.$$

**Solution** (2.3.37)**.** First note that

$$-1\le\cos\left(\frac{2}{x}\right)\le 1.$$

Since $x^4\ge 0$, by multiplying each part of the above inequality with $x^4$, we obtain

$$-x^4\le x^4\cos\left(\frac{2}{x}\right)\le x^4.$$

Since $\lim_{x\to 0} x^4=0$ and $\lim_{x\to 0} -x^4=0$, we conclude that

$$\lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right)=0$$

by the Squeeze Theorem.

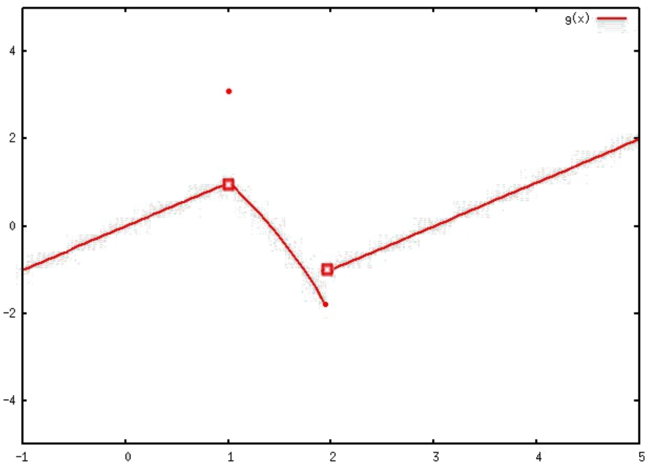


Figure 1: Graph of $g(x)$ for problem 2.3.48 (b).

**Solution** (2.3.48)**.**

(a)

(i)

$$\lim_{x\to 1-} g(x) = \lim_{x\to 1-} x = 1.$$

(ii) Note that

$$\lim_{x\to 1+} g(x) = \lim_{x\to 1+} 2 - x^2 = 2 - (1)^2 = 1.$$

Since the left and right limits are equal, we conclude that

$$\lim_{x\to 1} g(x) = 1.$$

(iii) $g(1) = 3$.

(iv)

$$\lim_{x\to 2-} g(x) = \lim_{x\to 2-} 2 - x^2 = 2 - (2)^2 = -2.$$

(v)

$$\lim_{x\to 2+} g(x) = \lim_{x\to 2+} x - 3 = 2 - 3 = -1.$$

(vi) Since the left and right limits are not equal, we conclude that

$$\lim_{x\to 2} g(x) \quad \text{does not exist.}$$

(b) See Fig.(1).

**Solution** (2.3.50)**.**

(a) See Fig.(2).

(b)

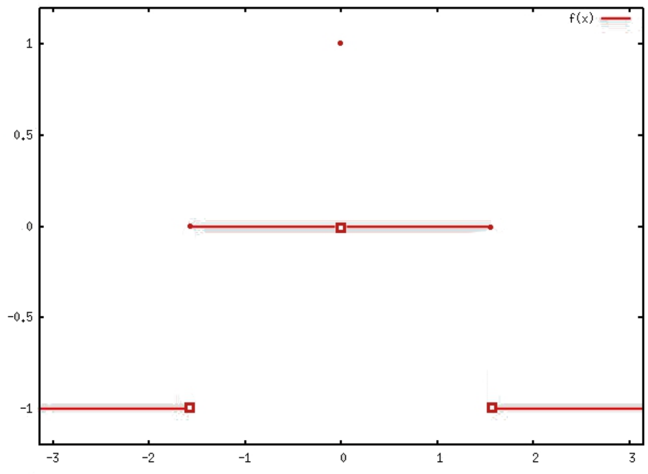


Figure 2: Graph of $f(x)$ for problem 2.3.50 (a).

(i)
$$\lim_{x\to 0} f(x) = 0.$$

(ii)
$$\lim_{x\to \pi/2-} f(x) = 0$$

(iii)
$$\lim_{x\to \pi/2+} f(x) = -1$$

(iv)
$$\lim_{x\to \pi/2} f(x) \quad \text{does not exist.}$$

(c) The limit $\lim_{x\to a} f(x)$ exists for all values $-\pi \leq a \leq \pi$ except for $-\pi/2$ and $\pi/2$.

**Solution** (2.3.55)**.** We have that

$$\lim_{x\to 1} f(x) - 8 = \lim_{x\to 1}\left[(f(x)-8)\cdot\frac{x-1}{x-1}\right] = \lim_{x\to 1}\left[(x-1)\cdot\frac{f(x)-8}{x-1}\right]$$
$$= \left[\lim_{x\to 1}(x-1)\right]\cdot\left[\lim_{x\to 1}\frac{f(x)-8}{x-1}\right] = 0\cdot 10 = 0.$$

Therefore

$$\lim_{x\to 1} f(x) = 8 + \lim_{x\to 1}(f(x)-8) = 8 + 0 = 8.$$

**Solution** (2.3.61)**.** Suppose that

$$\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2} = C$$

for some constant $C$. Then

$$\begin{aligned}\lim_{x\to-2}(3x^2+ax+a+3) &= \lim_{x\to-2}\left[(3x^2+ax+a+3)\cdot\frac{x^2+x-2}{x^2+x-2}\right]\\ &= \lim_{x\to-2}\left[(x^2+x-2)\cdot\frac{3x^2+ax+a+3}{x^2+x-2}\right]\\ &= \left[\lim_{x\to-2}(x^2+x-2)\right]\cdot\left[\lim_{x\to-2}\frac{3x^2+ax+a+3}{x^2+x-2}\right]\\ &= 0\cdot C = 0.\end{aligned}$$

Moreover, since

$$\lim_{x\to-2}(3x^2+ax+a+3) = 3(-2)^2+a(-2)+a+3 = 12-2a+a+3 = 15-a,$$

we must have that $15-a=0$, and therefore $a=15$. Hence

$$\begin{aligned}\lim_{x\to-2}\frac{3x^2+ax+a+3}{x^2+x-2} &= \lim_{x\to-2}\frac{3x^2+15x+18}{x^2+x-2} = \lim_{x\to-2}\frac{3(x+2)(x+3)}{(x+2)(x-1)}\\ &= \lim_{x\to-2}\frac{3(x+3)}{(x-1)} = \frac{3(-2+3)}{(-2-1)} = \frac{3}{-3} = -1.\end{aligned}$$

**Solution** (2.4.3)**.** Choose $\delta \le \min\{4-1.6^2, 2.4^2-4\} = \min\{1.44, 1.76\} = 1.44$.

**Solution** (2.4.5)**.** Choose $\delta \le \min\{\arctan(1.2)-\pi/2, \pi/2-\arctan(0.8)\} = 0.090660$.

**Solution** (2.4.19)**.** Let $\epsilon > 0$ and choose $\delta = 5\epsilon$. Then if $|x-3| < \delta$, we have that

$$\left|\frac{x}{5}-\frac{3}{5}\right| = \frac{1}{5}|x-3| < \frac{1}{5}\delta = \frac{1}{5}(5\epsilon) = \epsilon.$$

**Solution** (2.4.21)**.** Let $\epsilon > 0$ and choose $\delta = \epsilon$. Then if $|x-2| < \delta$, we have that

$$\left|\frac{x^2+x-6}{x-2}-5\right| = \left|\frac{(x+3)(x-2)}{x-2}-5\right| = |(x+3)-5| = |x-2| < \delta = \epsilon.$$

**Solution** (2.4.29)**.** Let $\epsilon > 0$ and choose $\delta = \sqrt{\epsilon}$. Then if $|x-2| < \delta$, we have that

$$|x^2-4x+5-1| = |x^2-4x+4| = |(x-2)^2| = |x-2|^2 < \delta^2 = \epsilon.$$

**Solution** (2.4.39)**.** Let $\epsilon = 1$. Then for any $\delta > 0$, let $N$ be an integer with $N > \sqrt{2}/\delta$. Set $x = \sqrt{2}/N$. Then $x$ is irrational, $|x| < \delta$, and $|f(x)-f(0)| = |1-0| = 1$. Thus the limit does not exist.

**Solution** (2.4.43)**.** Let $\Delta > 0$ and choose $\delta = e^{-\Delta}$. Then since $\ln(x)$ is strictly increasing, $0 < x < \delta$ implies that $\ln(x) < \ln(\delta) = \ln(e^{-\Delta}) = -\Delta$. Thus $\lim_{x\to0+}\ln(x) = -\infty$.

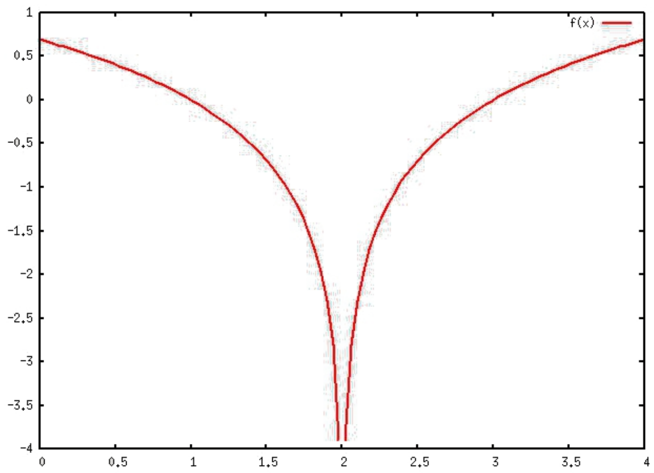


Figure 3: Graph of $f(x)$ for problem 2.5.15.

**Solution** (2.5.9)**.** Since $f$ and $g$ are continuous,

$$\begin{aligned}\lim_{x\to 3}[2f(x)-g(x)] &= \lim_{x\to 3} 2f(x) + \lim_{x\to 3} -g(x) = 2\lim_{x\to 3} f(x) - \lim_{x\to 3} g(x)\\ &= 2f(3) - g(3) = 2\cdot 5 - g(3) = 10 - g(3)\end{aligned}$$

Thus $10 - g(3) = 4$, from which it follows that $g(3) = 6$.

**Solution** (2.5.11)**.** To show that $f(x)$ is continuous at $x = -1$, we must show that $\lim_{x\to -1} f(x) = f(-1)$. The properties of limits tell us that

$$\begin{aligned}\lim_{x\to -1}(x+2x^3)^4 &= \left[\lim_{x\to -1}(x+2x^3)\right]^4 = \left[\lim_{x\to -1} x + \lim_{x\to -1} 2x^3\right]^4\\ &= \left[\lim_{x\to -1} x + 2\lim_{x\to -1} x^3\right]^4 = \left[(-1)+2(-1)^3\right]^4 = f(-1).\end{aligned}$$

Thus $f(x)$ is continuous at $x = -1$.

**Solution** (2.5.15)**.** The function $f(x)$ is discontinuous at $x = 2$, since $f(2) = \ln(0)$ is not defined. For a graph of the function, see Fig.(3).

**Solution** (2.5.19)**.** The function $f(x)$ is discontinuous at $x = 0$ because

$$\lim_{x\to 0} f(x) = 1,$$

but $f(0) = 0$. For a graph of the function, see Fig.(4).

**Solution** (2.5.27)**.** The domain of the function $G(t)$ is every value of $t$ for which $t^4 - 1 > 0$. Equivalently, this is when $t^4 > 1$ or more simply $t > 1$. The polynomial $t^4 - 1$, and the logarithm $\ln(x)$ are continuous by Theorem 2.5.7. Since $G$ is the composition of these two functions, it is continuous everywhere in its domain by Theorem 2.5.9.

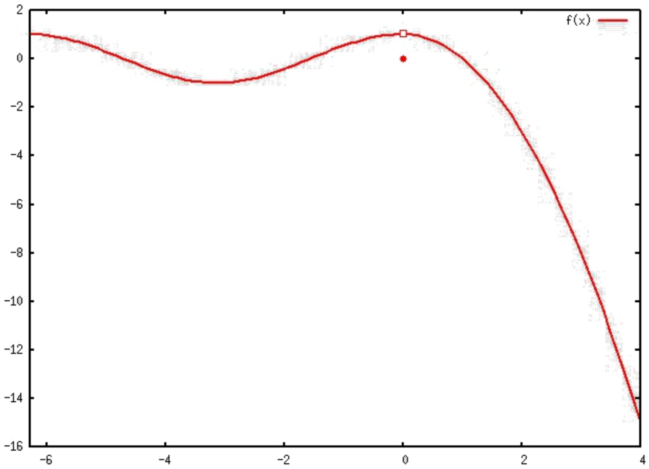


Figure 4: Graph of $f(x)$ for problem 2.5.19.

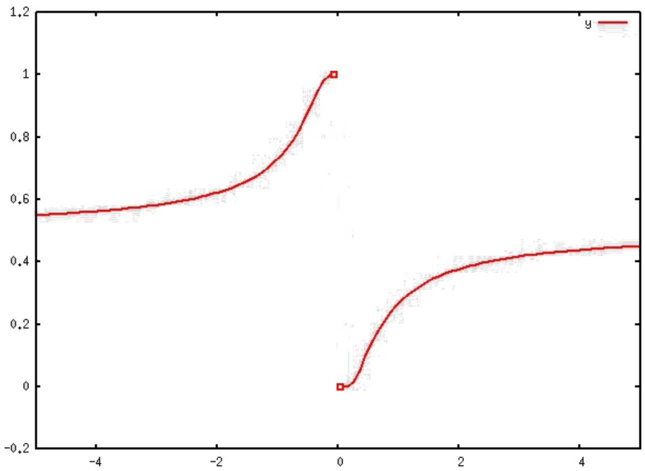


Figure 5: Graph of $y$ for problem 2.5.29.

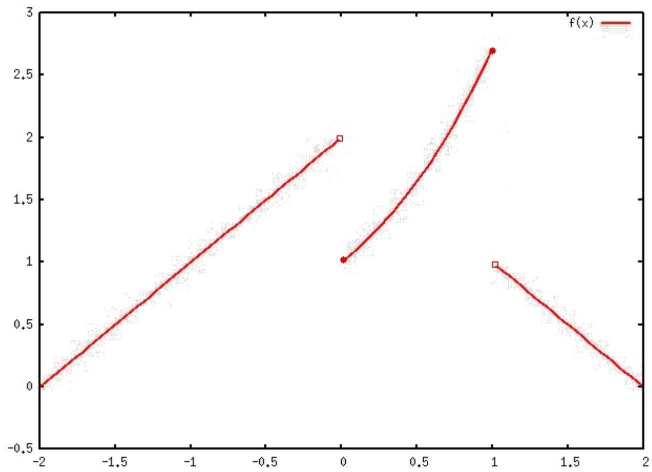


Figure 6: Graph of $y$ for problem 2.5.39.

**Solution** (2.5.29)**.** There is a discontinuity at $x = 0$. For a graph of the function, see Fig.(5).

**Solution** (2.5.33)**.** Since the exponential function $e^x$ and the polynomial $x^2 - x$ are continuous for all values of $x$, their composition $e^{x^2-x}$ is continuous. It follows that

$$\lim_{x\to 1} e^{x^2-x} = e^{(1)^2-(1)} = e^0 = 1.$$

**Solution** (2.5.39)**.** The function $f(x)$ is discontinuous at 0 and 1. It is continuous from the right at 0 and continuous from the left at 1. For a graph of the function , see Fig.(6).

**Solution** (2.5.41)**.** If $f(x)$ is to be continuous for all $x$, then in particular, we will require that the left and right limits of $f(x)$ at $x = 2$ are both equal to $f(2) = (2)^3 - c(2) = 8 - 2c$. Since

$$\lim_{x\to 2-} f(x) = \lim_{x\to 2-} cx^2 + 2x = c(2)^2 + 2(2) = 4c + 4,$$

and

$$\lim_{x\to 2+} f(x) = \lim_{x\to 2+} x^3 - cx = (2)^3 - c(2) = 8 - 2c,$$

this tell us that $8 - 2c = 4c + 4$. This means that $4 = 6c$, and therefore $c = 2/3$. For this value of $c$, $f(x)$ is continuous, since it is continuous at 2, and continuous everywhere else because it is defined to be polynomials elsewhere.

**Solution** (2.5.45)**.** To solve this problem, we apply the intermediate value theorem. The function $f(x)$ is the sum of the continuous functions $x^2$ and $10\sin(x)$, and is therefore continuous. Moreover $f(0) = 0$, $f(100\pi) = 10000\pi^2$, and $0 < 1000 < 10000\pi^2$. Thus by the intermediate value theorem, there is a value of $c$ with $0 < c < 100\pi$ such that $f(c) = 1000$.

**Solution** (2.5.49)**.** Define $f(x) = \cos(x) - x$. Since $f(x)$ is the difference of the continuous functions $\cos(x)$ and $x$, it is continuous. Moreover $f(\pi/6) = \cos(\pi/6) - \pi/6 = \frac{\sqrt{3}}{2} - \pi/6 > 0$ and $f(\pi/4) = \cos(\pi/4) - \pi/4 = \frac{\sqrt{2}}{2} - \pi/2 < 0$. Thus by the intermediate value theorem, there is a $c$ with $\pi/6 < c < \pi/4$ such that $f(c) = 0$. In particular, this shows that the equation $\cos(x) = x$ has a root in the interval $(0, 1)$.

**Solution** (2.5.59)**.** If is not continuous at any value of $x$. In fact, $\lim_{x\to a} f(x)$ does not exist for any $a$.

**Solution** (2.5.61)**.** The answer is yes. To see this, define $f(x) = x^3 + 1 - x$. Then $f$ is a polynomial, and is therefore continuous. Moreover $f(0) = 1$ and $f(-3) = (-3)^3 + 1 - (-3) = -27 + 1 + 3 = -23$. Thus by the intermediate value theorem, there is a $c$ with $-3 < c < 0$ such that $f(c) = 0$. In particular, this means that $c^3 + 1 - c = 0$, or rather $c^3 + 1 = c$. Thus $c$ is exactly 1 more than its cube.

**Solution** (2.5.63)**.** For values of $x \neq 0$, the function $1/x$ is continuous. Moreover $\sin(x)$ is continuous for all values of $x$, so the composition $\sin(1/x)$ must be continuous for all values of $x \neq 0$. Lastly $x^4$ is continuous for all values of $x$, so the product $x^4 \sin(1/x)$ must be continuous for all values of $x \neq 0$. Thus to prove that $f(x)$ is continuous for all $x$, it remains only to show that it is continuous at 0.

We note that $-1 \leq \sin(1/x) \leq 1$ and $x^4 \geq 0$, so therefore

$$-x^4 \leq x^4 \sin(1/x) \leq x^4.$$

Since

$$\lim_{x\to 0} -x^4 = \lim_{x\to 0} x^4 = 0,$$

the Squeeze theorem tells us that $\lim_{x\to 0} x^4 \sin(1/x) = 0$. Moreover, $f(0) = 0$ by definition. Thus $f(x)$ is continuous at $x = 0$.