BESICOVICH FUNCTIONS AND WEIGHTED ERGODIC THEOREMS FOR LCA GROUP ACTIONS

WILLIAM CASPER

Abstract. This is an abstract.

1. Introduction

Throughout this paper, we define G to be a locally compact group with (left) Haar measure λ(If G is compact, we assume the Haar measure is normalized, so that $\lambda(G) = 1$.). We denote by $C(G)$ the collection of all continuous, complex-valued functions on G, and by $C_{00}(G)$ the collection of all continuous, complex-valued functions on G with support contained in a compact set. Additionally, we denote by $C_0(G)$ the collection of all continuous, complex-valued functions on G which become arbitrarily small outside a compact set, i.e. given $\epsilon > 0$, there exists a compact set $C \subset G$ such that $|f(x)| < \epsilon$ for all $x \notin C$. For any linear space $\mathcal{F}$ of complex-valued functions on G, we denote by $\mathcal{F}^r$ and $\mathcal{F}^+$ the collection of all real-valued functions and positive functions in $\mathcal{F}$, respectively. Additionally, we use $\mathcal{F}^*$ to denote the dual space of $\mathcal{F}$. We denote by $\mathbf{M}(G)$ the collection of all complex measures on G.

1.1. Basic Definitions and Facts.

Definition 1. Let $\mathcal{F}$ be a linear space of complex-valued functions on G such that for every $f \in \mathcal{F}$, the function $x \mapsto f(a^{-1}x)$ $(x \mapsto f(xa^{-1}))$ is in $\mathcal{F}$ for all $a \in G$. Then $\mathcal{F}$ is called **left translation invariant** (**right translation invariant**). For any fixed $a \in G$, we define the operators

$$L(a) : f \mapsto L(a)f, \quad \text{where} \quad L(a)f : x \mapsto f(a^{-1}x);$$
$$R(a) : f \mapsto R(a)f, \quad \text{where} \quad R(a)f : x \mapsto f(xa^{-1}).$$

Given $a \in G$, the function $L(a) : f \mapsto L(a)f$ $(R(a) : f \mapsto R(a)f)$ are linear operators on any left (right) translation invariant space $\mathcal{F}$. The **left regular representation** L and **right regular representation** R on G are defined respectively by

$$L : a \mapsto L(a), \quad \text{where} \quad L(a) : f \mapsto L(a)f;$$
$$R : a \mapsto R(a), \quad \text{where} \quad R(a) : f \mapsto R(a)f.$$

Fact 2. The left and right regular representations are representations of G. The spaces $C(G)$, $C_{00}(G)$, and $C_0(G)$ are left and right translation invariant.

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Definition 3. Let $\mu \in \mathbf{M}(G)$ and $f \in L_p(G)$ for $1 \leq p \leq \infty$. We define the **convolution** of μ and f by

$$\mu * f(x) = \int_G f(y^{-1}x) d\mu(y).$$

The convolution of two measures $\mu, \nu \in \mathbf{M}(G)$, is the measure $\mu * \nu$ defined by

$$\mu * \nu(E) = \int_G \int_G 1_E(xy) d\nu(y) d\mu(x)$$

for all Borel sets E.

Fact 4. If $\mu \in \mathbf{M}(G)$ and $f \in L_p(G)$ for $1 \leq p \leq \infty$, then $\mu * f \in L_p$. For any $a \in G$, $\delta_a * f = L(a)f$. If $d\nu = f d\lambda$, then using Fubini's theorem and the left translation invariance of λ,

$$\begin{aligned} \mu * \nu(E) &= \int_G \int_G 1_E(xy) f(y) dy d\mu(x) = \int_G \int_G 1_E(y) f(x^{-1}y) dy d\mu(x) \\ &= \int_E \int_G f(x^{-1}y) d\mu(x) dy = \int_E \mu * f(y) dy. \end{aligned}$$

Thus $d(\mu * \nu) = \mu * f d\lambda$.

Definition 5. A bounded, continuous function f is **(weakly) almost periodic** if the set $\{L(a)f : a \in G\}$ is (weakly) conditionally compact in $C(G)$. The collection of all (weakly) almost periodic functions is denoted by (WAP(G)) AP(G).

Definition 6. A sequence $\{\mu_n\}_{n=1}^{\infty}$ **converges weakly to invariance** if for every $f \in C(G)$ and $t \in G$, we have

$$\int_G f(x^{-1}) - f(x^{-1}t) d\mu_n(x) = (\mu_n * f)(e) - (\mu_n * L(t)f)(e) = 0. \tag{1}$$

A sequence is **ergodic** if for every $f \in L_1(G)$, and every $t \in G$, we have

$$\lim_{n \to \infty} \|\mu_n * (f - L(t)f)\|_1 = 0. \tag{2}$$

Example 7. Let $\mu \in \mathbf{M}(G)$ and define $\{\mu_k\}_{k=1}^{\infty} \subset \mathbf{M}(G)$ by $\mu_1 = \mu$ and $\mu_{k+1} = \mu * \mu_k$ for $k \geq 1$. Let $f \in L_2(G)$. In general, the sequence of functions $\{\mu_k * f\}_{k=1}^{\infty}$ does not converge either pointwise or in $L_2(G)$. As an example, let G be a finite group and let S be a symmetric set of generators for G. The Haar measure λ on G is the normalized counting measure. Define a measure $\mu(A) = \lambda(A \cap S)/\lambda(A)$ for all subsets $A \subset G$. Then μ is a probability measure on G. If $e \in S$, then

$$\mu_k * f \to \frac{1}{|G|}$$

in $L_2(G)$ for all functions $f : G \to \mathbb{C}$. To see this, let $f : G \to C$ and let $\hat{G}$ represent the dual of G. For every equivalence class of representations $\sigma \in G$, fix a representation $\pi_\sigma \in \sigma$ with corresponding representation space H_σ of dimension d_σ and let $\{\xi_i^{(\sigma)}\}_{i=1}^{d_\sigma}$ be a fixed basis of H_σ. Define $\pi_{ij}^{(\sigma)}$ to be the coordinate map $\pi_{ij}^{(\sigma)}(x) = \langle \pi_\sigma(x)\xi_i^{(\sigma)}, \xi_j^{(\sigma)} \rangle$. Then $f \in L_2(G)$ and Peter-Weyl theorem tells us that

$$f(x) = \sum_{\sigma \in \hat{G}} \sum_{i,j=1}^{d_\sigma} f_{ij}^{(\sigma)} \sqrt{d_\sigma} \pi_{ij}^{(\sigma)}(x)$$

for some complex numbers $f_{ij}^{(\sigma)} \in \mathbb{C}$ satisfying

$$\|f\|_2 = \left(\sum_{\sigma \in \hat{G}} \sum_{i,j=1}^{d_\sigma} |f_{ij}^{(\sigma)}|^2 \right)^2 .$$

Since

$$\delta_a * \pi_{ij}^{(\sigma)}(x) = \pi_{ij}^{(\sigma)}(a^{-1}x) = \sum_{k=1}^{d_\sigma} \pi_{ik}^{(\sigma)}(a^{-1}) \pi_{kj}^{(\sigma)}(x)$$

It follows that

$$\begin{aligned}(\mu * f)(x) - \int_G f d\lambda = &\sum_{\sigma \in \hat{G}} \sum_{i,j=1}^{d_\sigma} f_{ij}^{(\sigma)} \sqrt{d_\sigma} \frac{1}{|S|} \sum_{s \in S \setminus \{e\}} \sum_{k=1}^{d_\sigma} \pi_{ik}^{(\sigma)}(s^{-1}) \pi_{kj}^{(\sigma)}(x) \\ &+ \sum_{\sigma \in \hat{G}} \sum_{i,j=1}^{d_\sigma} f_{ij}^{(\sigma)} \sqrt{d_\sigma} \frac{1}{|S|} \left(\pi_{ij}^{(\sigma)}(x) - \frac{|S|}{|G|} \sum_{y \in G} \pi_{ij}^{(\sigma)}(y) \right) .\end{aligned}$$

Therefore

$$\left\| (\mu * f)(x) - \int_G f d\lambda \right\|_2 \leq \left(\frac{|S| - 1}{|S|} \right) \|f\|_2 + \left(\frac{|G| - 1}{|G|} \right) \|f\|_2 + \left(\frac{1}{|S|} - \frac{1}{|G|} \right) \|f\|_2$$

In particular, it follows that the sequence$\{\mu_k\}_{k=1}^{\infty}$ is ergodic and converges weakly to invariance.

If $e \notin S$, then this is not necessarily the case. In particular, if $G = S_6$ and S consists of all two-cycles of G, then $\mu_k * 1_{\{e\}}$ does not converge. Instead, it cycles through a finite number of values. I do think it turns out to be ergodic, however. Details still need to be worked out.

2. Almost Periodic Functions

3. Convolution as a Markov Operator

Appendix A. Examples

Appendix B. Proofs of Theorems

References