INTRODUCTION TO SPECTRAL THEORY

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ABSTRACT. This is an abstract.

1. INTRODUCTION

Throughout this paper, we let H denote a Hilbert space, X a measureable space and Ω a σ-algebra of subsets of X. By an operator T on H we will always mean a linear transformation such that the operator norm

$$\|T\| := \sup\{\|T\psi\| : \psi \in H,\ \|\psi\| = 1\}$$

is finite. An operator will be called invertible if it has an algebraic inverse T^{-1} which is also an operator on H.

1.1. Basic Definitions and Facts.

Definition 1.1. The **spectrum** of a linear operator T on H is

$$\operatorname{spec}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not invertible. }\}. \tag{1}$$

The **approximate point spectrum** of T is

$$\operatorname{aspec}(T) = \{\lambda \in \mathbb{C} : \varrho(T - \lambda) = 0\}, \quad \text{where} \quad \varrho(T) = \inf\{\|T\psi\| : \psi \in H,\ \|\psi\| = 1\}. \tag{2}$$

Theorem 1.1. $\operatorname{aspec}(T) \subset \operatorname{spec}(T)$.

Theorem 1.2. If T is a normal operator, then $\operatorname{aspec}(T) = \operatorname{spec}(T)$.

Theorem 1.3 (Transforms of Spectra)**.**

(i) If $p \in \mathbb{C}[x]$, then

$$\operatorname{spec}(p(T)) = p(\operatorname{spec}(T)) := \{p(\lambda) : \lambda \in \operatorname{spec}(T)\}.$$

(ii) If T is invertible, then

$$\operatorname{spec}(T^{-1}) = (\operatorname{spec}(T))^{-1} := \{\lambda^{-1} : \lambda \in \operatorname{spec}(T)\}.$$

(iii) The spectrum of the adjoint of T satisfies

$$\operatorname{spec}(T^*) = (\operatorname{spec}(T))^* := \{\lambda^* : \lambda \in \operatorname{spec}(T)\}.$$

Theorem 1.4. Define

$$N_T(f) = \sup\{|f(\lambda)| : \lambda \in \operatorname{spec}(T)\}. \tag{3}$$

In general, $\operatorname{spec}(T)$ is a compact subset of the complex plane and $N_T(x) \leq \|T\|$. If T is Hermitian, then $\operatorname{spec}(T)$ is a subset of $\mathbb{R}$ and $N_T(p(x)) = \|p(T)\|$ for any polynomial $p \in \mathbb{R}[x]$.

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2. Spectral Measures

Definition 2.1. A **spectral measure** E on a measureable space (X, Ω) is a projection-valued (idempotent, hermitian) set function

$$E : \Omega \to \{ \text{ projections on } H\}$$

satisfying

(i) $E(X) = 1$;
(ii) for any collection of pairwise disjoint sets $\{A_k\}_{k=1}^{\infty}$,

$$E\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} E(A_k).$$

Theorem 2.1 (Properties of Spectral measures)**.** If E is a spectral measure, then for all $A, B \in \Omega$

(i) if $A \subset B$, then $E(A) \leq E(B)$;
(ii) if $A \subset B$, then $E(B \setminus A) = E(B) - E(A)$;
(iii) $E(A \cup B) + E(A \cap B) = E(A) + E(B)$;
(iv) $E(A \cap B) = E(A)E(B)$.

Theorem 2.2. A function

$$E : \Omega \to \{ \text{ projections on } H\}$$

is a spectral measure if and only if

(i) $E(X) = 1$;
(ii) for any two fixed elements $\psi, \phi \in H$,the function $\mu : \Omega \to \mathbb{C}$ defined by

$$\mu(A) = \langle E(A)\psi, \phi\rangle \quad \text{for all} \quad A \in \Omega$$

is a complex measure. We use the notation $d\langle E(\lambda)\psi, \phi\rangle = d\mu(\lambda)$ so that in general

$$\int 1_A d\langle E(\lambda)\psi, \phi\rangle = \langle E(A)\psi, \phi\rangle \quad \text{for all} \quad A \in \Omega.$$

Theorem 2.3. If E is a spectral measure and f is an E-measureable function, then there exists a unique operator denoted by either $\int f dE$ or $\int f(\lambda)dE(\lambda)$ and defined as

$$\left\langle \int f dE\psi, \phi \right\rangle = \int f(\lambda) d\langle E(\lambda)\psi, \phi\rangle. \tag{4}$$

Theorem 2.4 (Properties of $\int f dE$)**.** Given any E-measureable functions f, g and $\alpha \in \mathbb{C}$,

(i) $\int (\alpha f) dE = \alpha \int f dE$;
(ii) $\int (f + g) dE = \int f dE + \int g dE$;
(iii) $\left(\int f dE\right)^* = \int f^* dE$;
(iv) $\int fg dE = \left(\int f dE\right)\left(\int f dE\right)$.

Theorem 2.5. If E is a spectral measure and $E(A)$ commutes with T for every $A \in \Omega$, then $\int f dE$ commutes with T.

3. Complex Spectral Measures

For the remainder of the paper, we assume that X is a locally compact Hausdorff space and that Ω is the Borel σ-algebra on X.

Definition 3.1. A spectral measure is **regular** if for all $A \in \Omega$,

$$E(A) = \sup\{E(C) : C \subset A,\ C \text{ is compact }\}.$$

Definition 3.2. The **spectrum** of a spectral measure is

$$\operatorname{spec}(E) := X \setminus \{\lambda \in X : \lambda \in A,\ A \text{ is open},\ E(A) = 0\}.$$

A spectral measure is **compact** if its spectrum is compact.

Theorem 3.1. If E is a regular spectral measure, then $\operatorname{spec}(E)$ is closed and $E(X \setminus \operatorname{spec}(E)) = 0$ (and therefore $E(\operatorname{spec}(E)) = 1$).

Theorem 3.2. For any complex-valued, E-measureable function bounded on $\operatorname{spec}(E)$, define

$$N_E(f) = \sup\{|f(\lambda)| : \lambda \in \operatorname{spec}(E)\}.$$

Then if E is a compact, regular spectral measure and f is a continuous function on X, then $\left\|\int f dE\right\| = N_E(f)$.

Definition 3.3. A spectral measure is called **complex** when $X = \mathbb{C}$.

Theorem 3.3. Every complex spectral measure is regular.

Theorem 3.4. If E is a compact, complex spectral measure and if $T = \int \lambda dE(\lambda)$, then $\operatorname{spec}(T) = \operatorname{spec}(E)$.

Theorem 3.5. A complex spectral measure is defined completely by the operator $\int \lambda dE(\lambda)$. That is, given two complex spectral measures E_1 and E_2, $E_1 = E_2$ if and only if $\int \lambda dE_1(\lambda) = \int \lambda dE_2(\lambda)$.

Theorem 3.6. Let E be a complex spectral measure and T an operator. Then T commutes with $E(A)$ for all $A \in \Omega$ if and only if T commutes with both $\int \lambda dE(\lambda)$ and $\int \lambda^* dE(\lambda)$.

4. The Spectral Theorem

Theorem 4.1 (Spectral Theorem for Hermitian Operators)**.** Let T be a Hermitian operator. Then there exists a unique compact, complex spectral measure E such that $T = \int \lambda dE(\lambda)$.

Theorem 4.2 (Spectral Theorem for Normal Operators)**.** Let T be a normal operator. Then there exists a unique compact, complex spectral measure E such that $T = \int \lambda dE(\lambda)$.

Definition 4.1. For any normal operator T, we call the spectral measure E satisfying $T = \int \lambda dE(\lambda)$ the spectral measure of T.

Appendix A. Applications of the Spectral Theorem

A.1. **Weak Mixing.** As a first example application of the spectral theorem, we will use it to show that a measure-preserving transformation T is weak mixing when the only eigenfunctions of the unitary operator U_T defined by $U_Tf(x) = f(Tx)$ are the constants.

Definition A.1. Let (X, Ω, μ) be a probability space and T a measure preserving transformation (mpt) on X ($\mu(A) = \mu(T^{-1}(A))$ for all $A \in \Omega$). Then T is called weakly mixing if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |\mu(T^k(A \cap B) - \mu(A)\mu(B)| \to 0 \tag{5}$$

for all $A, B \in \Omega$.

Theorem A.1. A mpt T is weakly mixing if the only measureable eigenfunctions of U_T are the constants.

Proof. We first note that T is weakly mixing if and only if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle - \langle f, 1\rangle\langle 1, g\rangle| \to 0 \tag{6}$$

for all $f, g \in L_2(\mu)$.

Let V be the closed linear subspace of the eigenfunctions of T in $L_2(\mu)$ and let E be the spectral measure of T. Then for any $\lambda_0 \in \text{spec}(U_T)$, we have that

$$U_TE(\{\lambda_0\}) = \int \lambda dE(\lambda) \int 1_{\lambda_0}(\lambda)dE(\lambda) = \int \lambda 1_{\{\lambda_0\}}(\lambda)dE(\lambda) = \lambda_0 E(\{\lambda_0\}),$$

so that in particular $U_TE(\{\lambda_0\})f = \lambda_0 f$ for every $f \in L_2(\mu)$. Thus $E(\{\lambda_0\})f \in V$ for all $f \in L_2(\mu)$. If $f \in V^\perp$, this implies that

$$0 = \langle E(\{\lambda_0\})f, f\rangle = \langle E(\{\lambda_0\})^2 f, f\rangle = \langle E(\{\lambda_0\})f, E(\{\lambda_0\})f\rangle,$$

and therefore $E(\{\lambda_0\})f = 0$.

Now fix an $f \in V^\perp$ and $g \in L_2(\mu)$ and define μ to be the complex Borel measure on the spectrum of T satisfying $d\mu = d\langle E(\lambda)f, g\rangle$. Then for all $\lambda_0 \in \text{spec}(T)$, we have that

$$\mu(\{\lambda_0\}) = \langle E(\{\lambda\})f, g\rangle = 0.$$

Setting $\Delta = \{(\lambda, \omega) \in \text{spec}(U_T) \times \text{spec}(U_T) : \lambda = \omega\}$, we find

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle|^2 &= \frac{1}{n}\sum_{k=0}^{n-1} |\langle \int \lambda^k dE(\lambda)f, g\rangle|^2 = \frac{1}{n}\sum_{k=0}^{n-1} |\lambda^k \int \langle dE(\lambda)f, g\rangle|^2 \\
&= \frac{1}{n}\sum_{k=0}^{n-1} |\int \lambda^k d\mu(\lambda)|^2 = \frac{1}{n}\sum_{k=0}^{n-1} \int \lambda^k d\mu(\lambda) \int (\overline{\omega})^k d\mu^*(\omega) \\
&= \frac{1}{n}\sum_{k=0}^{n-1} \int\int (\lambda\overline{\omega})^k d\mu(\lambda)d\mu^*(\omega) = \int \frac{1}{n}\sum_{k=0}^{n-1} (\lambda\overline{\omega})^k d(\mu \times \mu^*)(\lambda, \omega) \\
&= \int_{\Delta^c} \frac{1}{n}\frac{1 - (\lambda\overline{\omega})^k}{1 - \lambda\overline{\omega}} d(\mu \times \mu^*)(\lambda, \omega) + \int_\Delta 1 d(\mu \times \mu^*)(\lambda, \omega).
\end{aligned}$$

Additionally,

$$\int_{\Delta} 1 d(\mu \times \mu^*)(\lambda, \omega) = \int \int 1_{\Delta}(\lambda, \omega) d\mu(\lambda) d\mu^*(\omega) = \int \mu(\{\omega\}) d\mu^*(\omega) = 0.$$

For $(\lambda, \omega) \notin \Delta$, $(1 - (\lambda\overline{\omega})^k)/(1 - \lambda\overline{\omega})$ is a cyclotomic polynomial, and is therefore bounded on the compact set $\operatorname{spec}(U_T)$. Thus by the bounded convergence theorem

$$\lim_{n\to\infty} \int_{\Delta^c} \frac{1}{n} \frac{1 - (\lambda\overline{\omega})^k}{1 - \lambda\overline{\omega}} d(\mu \times \mu^*)(\lambda, \omega) = \int_{\Delta^c} 0 d(\mu \times \mu^*)(\lambda, \omega) = 0.$$

We conclude that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle|^2 = 0.$$

By the Cauchy-Schwartz inequality,

$$\left(\sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle| \right)^2 \leq n \sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle|^2.$$

Dividing both sides by n^2, we find

$$\left(\frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle| \right)^2 \leq \frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle|^2,$$

and therefore

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle| = 0.$$

Now for any $f, g \in L_2(G)$, if the eigenfunctions of U_T are the constants, $f - \langle f, 1\rangle \in V^\perp$. Therefore

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k (f - \langle f, 1\rangle), g\rangle| &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k f - \langle f, 1\rangle, g\rangle| \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} |\langle U_T^k f, g\rangle - \langle f, 1\rangle\langle 1, g\rangle| = 0. \end{aligned}$$

Therefore T is weakly mixing. □

A.2. **Almost Periodic Functions.**

Appendix B. Proofs of Theorems

B.1. **Proof for Section 1.**

Proof of Theorem (1.1). If $\lambda \notin \operatorname{spec}(T)$, then $T - \lambda$ is invertible. Thus for any $\psi \in H$ with $\|\psi\| = 1$, we have that

$$1 = \|\psi\| = \|(T - \lambda)^{-1}(T - \lambda)\psi\| \leq \|(T - \lambda)^{-1}\| \cdot \|(T - \lambda)\psi\|.$$

Thus ϱ (as defined by Eq. (1)) satisfies $\varrho(T - \lambda) \geq 1/\|(T - \lambda)^{-1}\|$ and so $\lambda \notin \operatorname{aspec}(T)$. This proves our theorem. □

Lemma B.1. An operator T is invertible if and only if its range is dense in H and there exists a positive real number $c > 0$ such that $\|T\psi\| \geq c\|\psi\|$ for all $\psi \in H$.

Proof. If T is invertible, then it is a bijection and therefore the range must be H. Moreover,

$$\|\psi\| = \|T^{-1}T\psi\| \leq \|T^{-1}\| \cdot \|T\psi\|,$$

so $\|T\psi\| \geq c\|\psi\|$ with $c = 1/\|T^{-1}\|$.

Conversely, suppose the range of T is dense in H and there exists a positive real number $c > 0$ such that $\|T\psi\| \geq c\|\psi\|$ for all $\psi \in H$. We first show that the range of T is H. Let $\{\phi_i\}_{i=1}^{\infty}$ be a convergent sequence in the range of H converging to ϕ. For every $i > 0$, there exists a $\psi_i \in H$ such that $T\psi_i = \phi_i$. Moreover, $\|\psi_i - \psi_j\| \leq \|\phi_i - \phi_j\|/c$. It follows that the sequence $\{\psi_i\}_{i=1}^{\infty}$ is Cauchy and therefore converges to a function $\psi \in H$. Since T is continuous, $\phi = T\psi$, and therefore ϕ is in the range of T. We conclude that the range of T is closed. Since the range of T is dense in H, the range of T must be H.

The kernel of T is trivial, since if $\psi \in \ker(T)$, then $\|\psi\| \leq \|T\psi\|/c = 0$, implying that $\psi = 0$. Thus T is a bijection, and all that is left to show is that the algebraic inverse, which we call T^{-1}, is bounded. We have that $\|T^{-1}\psi\| \leq \|TT^{-1}\psi\|/c \leq \|\psi\|/c$. It follows that $\|T^{-1}\| \leq 1/c$. This proves our theorem. Incidentally, this also shows us that $c = 1/\|T^{-1}\|$ is the "sharpest" value for c. □

Proof of Theorem (1.2). By Theorem (1.1), we need only prove $\text{spec}(T) \subset \text{aspec}(T)$. Suppose that $\lambda \notin \text{aspec}(T)$. Then there exists a constant $c > 0$ such that $\|(T - \lambda)\psi\| \geq c\|\psi\|$ for all $\psi \in H$. By Lemma (B.1), we need only show that the range of $T - \lambda$ is dense in H. Since T commutes with T^*, $T - \lambda$ commutes with $(T - \lambda)^* = T^* - \lambda^*$, and it follows that $\|(T - \lambda)\psi\| = \|(T^* - \lambda^*)\psi\|$ for all $\psi \in H$. If $\phi \in \text{rangle}(T - \lambda)^{\perp}$, then $0 = \langle (T - \lambda)\psi, \phi\rangle = \langle \psi, (T^* - \lambda^*)\phi\rangle$ for all $\psi \in H$, and therefore $(T^* - \lambda^*)\phi = 0$. It follows that $\phi = 0$, since $\|\phi\| \leq \|(T - \lambda)\phi\|/c = \|(T^* - \lambda^*)\phi\|/c = 0$. Thus $\text{rangle}(T - \lambda)^{\perp} = \{0\}$ and it follows that $\text{rangle}(T - \lambda)$ is dense in H. This proves our theorem. □

Proof of Theorem (1.3). (i) Let $p \in \mathbb{C}[x]$ and $\lambda \in \text{spec}(T)$. Then λ is a root of $r(x) = p(x) - p(\lambda)$ and therefore there exists a polynomial $q \in \mathbb{C}[x]$ such that $q(x)(x - \lambda) = p(x) - p(\lambda)$. If $r(T)$ is invertible, then $q(T)$ commutes with $r^{-1}(T)$ and

$$\begin{aligned}(T - \lambda)q(T)r(T)^{-1} &= r(T)r(T)^{-1} = 1 = r(T)^{-1}r(T) = r(T)^{-1}(T - \lambda)q(T) \\ &= r(T)^{-1}q(T)(T - \lambda) = q(T)r(T)^{-1}(T - \lambda).\end{aligned}$$

It follows that $(T - \lambda)$ is invertible with $(T - \lambda)^{-1} = q(T)r(T)^{-1}$, which is a contradiction. Thus $r(T)$ is not invertible and $p(\lambda) \in \text{spec}(p(T))$.

Conversely, suppose $\lambda \in \text{spec}(p(T))$ and let $\{r_i\}_{i=1}^{n}$ be the roots of the polynomial $p(x) - \lambda$. We have that $p(x) - \lambda = (x - r_1) \dots (x - r_n)$ and therefore $p(T) - \lambda = (T - r_1) \dots (T - r_n)$. Since $p(T) - \lambda$ is not invertible, $(T - r_j)$ is not invertible for some j. Whence $r_j \in \text{spec}(T)$ and $p(r_j) - \lambda = 0$. We conclude that $\lambda \in p(\text{spec}(T))$. This proves (i).

(ii) Note that for any $\lambda \in \mathbb{C}$, we have that $T^{-1} - \lambda^{-1} = -T^{-1}\lambda^{-1}(T - \lambda)$, and it follows that $T^{-1} - \lambda^{-1}$ is invertible if and only if $T - \lambda$ is invertible. This proves (ii).

(iii) If $\lambda \notin \text{spec}(T)$, then $T - \lambda$ is invertible. It follows that $(T - \lambda)^* = T^* - \lambda^*$ is invertible, and therefore $\text{spec}(T^*) \subset \text{spec}(T)^*$. By the same argument with T replaced by T^*, $\text{spec}(T) = \text{spec}((T^*)^*) \subset \text{spec}(T^*)^*$, and therefore $\text{spec}(T)^* \subset (\text{spec}(T^*)^*)^* = \text{spec}(T^*)$. This proves our theorem.

□

Lemma B.2. If T is an operator such that $\|1-T\|<1$, then T is invertible.

Proof. Define $c>0$ by $c=1-\|1-T\|$. Then

$$\|T\psi\| = \|\psi-(\psi-T\psi)\| \geq \|\psi\|-\|(1-T)\psi\| \geq \|\psi\|-\|(1-T)\|\cdot\|\psi\| = c\|\psi\|.$$

Thus by Lemma (B.1), we need only show that the range of T is dense in H. Let $\phi\in H$ and let $\delta=\inf\{\|T\psi-\phi\| : \psi\in H\}$. Suppose that $\delta>0$. Then for all $\epsilon=\delta\frac{c}{1-c}$, there exists $\psi\in H$ such that $\delta\leq\|T\psi-\phi\|<\delta+\epsilon$. Moreover

$$\delta \leq \|T(T\psi-\phi)-(T\psi-\phi)\| = \|(1-T)(T\psi-\phi)\| < (1-c)\|T\psi-\phi\| = (1-c)(\delta+\epsilon) \leq \delta.$$

That is, $\delta<\delta$, which is a contradiction. We conclude that $\delta=0$. Since $\phi\in H$ was taken arbitrarily, this means that the range of T is dense in H. This proves our lemma. □

Proof of Theorem (1.4). If $\lambda_0\notin\operatorname{spec}(T)$, then $T-\lambda_0$ is invertible. If $\lambda\in\mathbb{C}$ with $|\lambda-\lambda_0|<r:=1/\|(T-\lambda_0)^{-1}\|$, then

$$\begin{aligned}\|1-(T-\lambda_0)^{-1}(T-\lambda)\| &= \|(T-\lambda_0)^{-1}[(T-\lambda_0)-(T-\lambda)]\| \\ &\leq \|(T-\lambda_0)^{-1}\|\cdot|\lambda-\lambda_0|<1.\end{aligned}$$

Therefore by Lemma (B.2) $(T-\lambda_0)^{-1}(T-\lambda)$ is invertible and it follows that $(T-\lambda)$ must be invertible. We conclude that the ball $B(\lambda_0;r)$ about λ_0 of radius r is contained in $\mathbb{C}\setminus\operatorname{spec}(T)$. It follows that $\mathbb{C}\setminus\operatorname{spec}(T)$ is open and therefore $\operatorname{spec}(T)$ is closed. Moreover, if $\lambda\in\mathbb{C}$ satisfies $\|T\|<|\lambda|$, then $\|1-(1-T/\lambda)\|=\|T/\lambda\|<1$ and therefore $1-T/\lambda$ is invertible by Lemma(B.2). It follows that $T-\lambda$ is invertible, and therefore $\lambda\notin\operatorname{spec}(T)$. Thus if $\lambda\in\operatorname{spec}(T)$, then $|\lambda|\leq\|T\|$ necessarily. In particular, this shows that $N_T(x)\leq\|T\|$ and that $\operatorname{spec}(T)$ is a closed and bounded subset of $\mathbb{C}$ (and therefore compact).

Suppose T is Hermitian and $\lambda\in\operatorname{spec}(T)$. Then T is normal and $\operatorname{spec}(T)=\operatorname{aspec}(T)$ by Theorem (1.2). Thus there exists a sequence $\{\psi_i\}_{i=1}^{\infty}\subset H$ such that $\|\psi_i\|=1$ for all i and $\|(T-\lambda)\psi_i\|\to 0$. Thus

$$\begin{aligned}|\lambda-\lambda^*| &= |\lambda-\lambda^*|\cdot\|\psi_i\|^2 = |\langle(T-\lambda)\psi_i,\psi_i\rangle-\langle(T-\lambda^*)\psi_i,\psi_i\rangle| \\ &= |\langle(T-\lambda)\psi_i,\psi_i\rangle-\langle\psi_i,(T-\lambda)\psi_i\rangle| \\ &\leq 2\|(T-\lambda)\psi_i\|\cdot\|\psi_i\| = 2\|(T-\lambda)\psi_i\|\to 0.\end{aligned}$$

It follows that λ is real. Moreover for any $\lambda\in\mathbb{R}$, since T is Hermitian, we have the relation

$$\begin{aligned}\|T^2\psi-\lambda^2\psi\|^2 &= \langle T^2\psi-\lambda^2\psi, T^2\psi-\lambda^2\psi\rangle \\ &= \|T^2\psi\|^2+|\lambda|^4\|\psi\|^2-(\lambda^2)^*\langle T^2\psi,\psi\rangle-\lambda^2\langle\psi,T^2\psi\rangle \\ &= \|T^2\psi\|^2+\lambda^4\|\psi\|^2-2\lambda^2\|T\psi\|^2.\end{aligned}$$

Now let $\{\psi_i\}_{i=1}^{\infty}\subset H$ be a sequence such that $\|\psi_i\|=1$ for all i and $\|T\psi_i\|\to\|T\|$. Then taking $\lambda=\|T\|$ in the above relation, we find that

$$\begin{aligned}\|(T^2-\|T\|^2)\psi_i\|^2 &= \|T^2\psi_i-\lambda^2\psi_i\|^2 = \|T^2\psi_i\|^2+\lambda^4\|\psi_i\|^2-2\lambda^2\|T\psi_i\|^2 \\ &= \|T^2\psi_i\|^2+\|T\|^4-2\|T\|^2\|T\psi_i\|^2\to 0.\end{aligned}$$

Thus $\|T\|^2\in\operatorname{spec}(T^2)$, and it follows from Theorem (1.3) that either $\|T\|\in\operatorname{spec}(T)$ or $-\|T\|\in\operatorname{spec}(T)$. In particular, this proves $N_T(x)=\|T\|$. If $p(x)\in\mathbb{R}[x]$, then

$p(T)$ is Hermitian and therefore $N_T(p(x)) = N_{p(T)}(x) = \|p(T)\|$. This proves our theorem. □

B.2. **Proofs for Section 2.**

B.3. **Proofs for Section 3.**

B.4. **Proofs for Section 4.**

Lemma B.3 (Weierstrass Approximation Theorem)**.** Let X be a compact subset of $\mathbb{R}$ and let f be a continuous function on X. Then there exists a sequence of real polynomials $\{p_i\}_{i=1}^{\infty}$ such that $p_i \to f$ uniformly on X.

Lemma B.4. Let L be a bounded linear functional on $R[x]$ and let X be a compact subset of R. Then there exists a unique Borel measure μ on X satisfying

$$L(p) = \int p(\lambda) d\mu(\lambda) \quad \text{for all} \quad p \in \mathbb{R}[x].$$

Sketch of proof. Let Ω be the collection of all Borel subsets of X and let $A \in \Omega$. Let $\{p_i\}_{i=1}^{\infty} \subset \mathbb{R}[x]$ be a sequence of polynomials with $p_i \to 1_A$ uniformly on X. Define $\mu(A)$ by

$$\mu(A) = \lim_{i\to\infty} L(p_i).$$

Then μ is a well-defined complex Borel measure on X. □

Proof of Theorem (4.1). Let $\psi, \phi \in H$ and define a function

$$L : \mathbb{R}[x] \to \mathbb{C}$$

by $L(p) = \langle p(T)\psi, \phi\rangle$. Then L is linear and

$$|L(p)| \leq \|p(T)\psi\| \cdot \|\phi\| \leq \|p(T)\| \cdot \|\psi\| \cdot \|\phi\| \leq N_T(p(x)) \cdot \|\psi\| \cdot \|\phi\|$$

and therefore L is a linear functional on $\mathbb{R}[x]$. The set $\mathbb{R}[x]$ is a dense subset of the collection of all continuous, real-valued functions on $\text{spec}(T)$, and it follows that there exists a unique complex measure μ on $X = \text{spec}(T)$ with σ-algebra Ω consisting of all Borel subsets of $\text{spec}(T)$ such that $L(p) = \int p(\lambda)d\mu(\lambda)$ for all $p \in \mathbb{C}[x]$. For given $\psi, \phi \in H$, we denote this measure by $\mu_{(\psi,\phi)}$. Let $\psi_1, \psi_2, \phi_1, \phi_2 \in H$ and let $\alpha \in \mathbb{C}$.

$$\begin{aligned}\int p(\lambda) d\mu_{(\psi_1+\psi_2,\phi)}(\lambda) &= \langle p(T)(\psi_1+\psi_2), \phi\rangle = \langle p(T)\psi_1, \phi\rangle + \langle p(T)\psi_2, \phi\rangle \\ &= \int p(\lambda) d\mu_{(\psi_1,\phi)}(\lambda) + \int p(\lambda) d\mu_{(\psi_2,\phi)}(\lambda),\end{aligned}$$

from which it follows that

$$\mu_{(\psi_1+\psi_2,\phi)} = \mu_{(\psi_1,\phi)} + \mu_{(\psi_2,\phi)}.$$

Similarly,

$$\mu_{(\psi,\phi_1+\phi_2)} = \mu_{(\psi,\phi_1)} + \mu_{(\psi,\phi_2)};$$
$$\mu_{(\alpha\psi,\phi)} = \alpha\mu_{(\psi,\phi)} \quad \text{and} \quad \mu_{(\psi,\alpha\phi)} = \alpha^*\mu_{(\psi,\phi)}.$$

Lastly, for $A \in \Omega$ we have that

$$\begin{aligned}|\mu_{(\psi,\phi)}(A)| \leq |\mu_{(\psi,\phi)}|(X) &= \sup\left\{\frac{1}{N_T(p)}\left|\int p d\mu_{(\psi,\phi)}\right| : p \in \mathbb{C}[x]\right\} \\ &= \sup\{|\langle p(T)\psi, \phi\rangle|/N_T(p) : p \in \mathbb{C}[x]\} = \|\psi\| \cdot \|\phi\|.\end{aligned}$$

For any $A \in \Omega$, we define $\mu_A(\psi,\phi) = \mu_{(\psi,\phi)}(A)$. The above properties show us that μ_A is a symmetric, bilinear functional, and therefore for every $A \in \Omega$, there exists a unique Hermitian operator $E(A)$ such that $\mu_A(\psi,\phi) = \langle E(A)\psi,\phi\rangle$ for all $\psi,\phi \in H$.

We first show that $E(A)$ is idempotent for all $A \in \Omega$ by proving the more general result $E(A\cap B) = E(A)E(B)$ for all $A, B \in \Omega$. Fix $B \in \Omega$ and let $\{q_i\}_{i=1}^{\infty} \subset \mathbb{R}[x]$ be a fixed sequence of polynomials with $q_i(\lambda) \to 1_A(\lambda)$ uniformly on X. Also fix $\psi,\phi \in H$. For each i, define a measure ν_i by $d\nu_i(\lambda) = q_i(\lambda)d\mu_{(\psi,\phi)}(\lambda)$. Then for any $p \in \mathbb{C}[x]$, $q(T)$ is Hermitian commutes with $p(T)$ and we have that

$$\int p(\lambda)d\nu_i(\lambda) = \int p(\lambda)q_i(\lambda)d\mu_{(\psi,\phi)}(\lambda) = \langle p(T)q_i(T)\psi,\phi\rangle$$
$$= \langle p(T)\psi, q_i(T)\phi\rangle = \int p(\lambda)d\mu_{(\psi,q_i(T)\phi)}.$$

Let $A \in \Omega$ and $\{p_i\}_{i=1}^{\infty} \subset \mathbb{R}[x]$ be a sequence of polynomials with $p_i(\lambda) \to 1_A(\lambda)$ uniformly on X. Then the dominated convergence theorem tells us that

$$\nu_i(A) = \lim_{i\to\infty}\int p_i(\lambda)q_i(\lambda)d\mu_{(\psi,\phi)}(\lambda) = \lim_{i\to\infty}\int p_i(\lambda)d\mu_{(\psi,q_i(T)\phi)}(\lambda)$$
$$= \int 1_A(\lambda)d\mu_{(\psi,q_i(T)\phi)}(\lambda) = \mu_A(\psi,q_i(T)\phi) = \langle E(A)\psi, q_i(T)\phi\rangle$$
$$= \langle q_i(T)E(A)\psi,\phi\rangle = \int q_i(\lambda)d\mu_{(E(A)\psi,\phi)}(\lambda)$$

The dominated convergence theorem also tells us that

$$\langle E(A\cap B)\psi,\phi\rangle = \int 1_{A\cap B}(\lambda)d\mu_{(\psi,\phi)}(\lambda) = \int 1_A(\lambda)1_B(\lambda)d\mu_{(\psi,\phi)}(\lambda)$$
$$= \lim_{i\to\infty}\nu_i(A) = \lim_{i\to\infty}\int q_i(\lambda)d\mu_{(E(A)\psi,\phi)}(\lambda)$$
$$= \int 1_B(\lambda)d\mu_{(E(A)\psi,\phi)}(\lambda) = \langle E(B)E(A)\psi,\phi\rangle.$$

Since $A, B \in \Omega$ and $\psi,\phi \in H$ were arbitrary, this proves that $E(A\cap B) = E(A)E(B)$ for all $A, B \in \Omega$. Thus E is idempotent.

Lastly, we have that

$$\langle E(X)\psi,\phi\rangle = \mu_X(\psi,\phi) = \mu_{(\psi,\phi)}(X) = \int 1d\mu_{(\psi,\phi)}(\lambda) = \langle\psi,\phi\rangle,$$

and by Theorem (2.2) this means that $E(X)$ is a compact, complex spectral measure. A quick calculation shows that

$$\int \lambda d\mu_{(\psi,\phi)}(\lambda) = \langle T\psi,\phi\rangle.$$

The uniqueness of the measure follows from Theorem (3.5). This proves our theorem. □

Sketch of proof of Theorem (4.2). Let T be normal and define T_1 and T_2 by $T_1 = \frac{1}{2}(T+T^*)$ and $T_2 = \frac{1}{2i}(T-T^*)$. Then T_1 and T_2 are Hermitian with $T = T_1 + iT_2$ and there exist unique spectral measures E_1, E_2 such that $T_i = \int \lambda dE_i(\lambda)$ for $i = 1, 2$.

Define $\mathcal{A} = \{A + iB : A, B \text{ real Borel sets }\}$ and define a projection-valued set function E by $E(A+iB) = E_1(A)E_2(B)$. Then $\mathcal{A}$ is an algebra of sets and the

σ-algebra generated by $\mathcal{A}$ is Ω, the collection of all Borel subsets of $\mathbb{C}$. Moreover, $E(\mathbb{C}) = E_1(\mathbb{R})E_2(\mathbb{R}) = 1$ and E extends uniquely to a projection-valued measure E on $\mathbb{C}$.

□