REPRESENTATION THEORY AND THE FOURIER-STIELTJES TRANSFORM FOR COMPACT GROUPS

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ABSTRACT

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In this paper, we set up the basic definitions of the theory representations of locally compact topological groups. As necessary background material we define the normed linear space $\mathbf{M}(G)$ of all complex measures on G and show that there is a linear isometry between this space and the dual of the space $C_0(G)$ of all continuous functions on G going to zero at infinity. Using this relationship, we then define the convolution and adjoint of complex measures on G. We then discuss the theory of representations of a locally compact group G and in particular explain a means of "extending" representations of G to representations of $\mathbf{M}(G)$ and "restricting" representations of $\mathbf{M}(G)$ to representations of G. We use this to obtain the Gel'fand-Raĭkov theorem which shows that the collection of all irreducible, continuous unitary representations of a locally compact group "separates points". This is then used to prove the Peter-Weyl theorem which establishes that the collection of coordinate of a compact group forms an orthonormal basis. The paper culminates with the definition of the Fourier-Stieltjes transform, as defined in [1].

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CHAPTER 1. COMPLEX MEASURES ON G

1.1. Introduction and Notation

This chapter is an introduction to the theory of complex measures generated by linear functionals of the Banach space $C_0(X)$ of all continuous, complex-valued functions on a locally compact Hausdorff space X vanishing at infinity. Throughout this chapter, we assume that X is a locally compact Hausdorff space and that G is a locally compact T_0 topological group. Thus results obtained for X will be able to be applied immediately to G[1]. We will let $C_0(X)$ denote the collection of all complex-valued functions on X that are arbitrarily small outside a compact set, $C_{00}(X)$ denote the collection of all complex-valued functions with support contained in a compact set, and $\mathfrak{M}$ the collection of all lower semicontinuous functions on X. Given any linear space of complex-valued functions S, we denote by S^* the collection of all bounded linear functionals (the dual space) on S. Additionally, we use S^r and S^+ to denote the collections of all real and positive functions. For $f, g \in S^r$, we define the functions $\min(f, g)$ and $\max(f, g)$ (which may or may not be in S) by

$$\min(f,g)(x) = \frac{1}{2}(f(x) + g(x) - |f(x) - g(x)|) = \begin{cases} f(x), & f(x) \leqslant g(x); \\ g(x), & \text{otherwise} \end{cases}, \qquad (1)$$

and

$$\max(f,g)(x) = \frac{1}{2}(f(x) + g(x) + |f(x) - g(x)|) = \begin{cases} f(x), & f(x) \geqslant g(x); \\ g(x), & \text{otherwise} \end{cases}. \qquad (2)$$

One should be careful not to confuse this later with the definitions of $\max(T_1, T_2)$ and $\min(T_1, T_2)$ for linear functionals. We will us $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ to denote the real and

[1]All T_0 locally compact topological groups are Hausdorff. See [2] for details.

imaginary components of a function $f \in S$

Definition 1. A *topological group* is a group G with a topology under which the group operations

(i) $G \times G \to G : (x, y) \mapsto xy$

(ii) $G \to G : x \mapsto x^{-1}$

are continuous. A topological group G is called *locally compact* if it is locally compact as a topological space. That is, if every member $x \in G$ has an open neighborhood contained in a compact set.

We will accept without proof various properties of locally compact topological groups, including the existence of a (left or right) Haar measure. Unless otherwise stated, λ will always denote a left Haar measure on G. If G is compact, we will choose λ to be the unique Haar measure such that $\lambda(G) = 1$. The symbols $\tau : G \times G \to G$ and $\theta : G \to G$ defined by $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ will always be used to denote the continuous mappings of multiplication and inversion on G, unless stated otherwise.

In this chapter we will establish a method for constructing a complex measure from a linear functional $T \in C_0^*(X)$ and obtain a linear isomorphism $\Phi : C_0^*(X) \to \mathbf{M}(X)$. The linear space $\mathbf{M}(X)$ can be given a norm defined via the total variation $|\mu|$ of a measure $\mu \in \mathbf{M}(X)$. In particular, the norm is defined by $\|\mu\| = |\mu|(X)$. Under this definition, the mapping Φ is an isometry. During the second half of the chapter, we consider the specific case where $X = G$. We will define two additional natural operations for the space $C_0^*(G)$: convolution and adjoint. Via the transformation Φ, these operations will be extended to operations on $\mathbf{M}(G)$. In particular, defining these operations will allow us to consider $\mathbf{M}(G)$ as a $*$-algebra, a fact that will play a central role in the construction of continuous, unitary representations of locally compact groups later on.

1.2. Properties of Complex Measures

In this section, we establish some results for complex measures necessary for the rest of the chapter, as well as for some results in future chapters. In particular, we will define the total variation of a complex measure and use it to define a norm on $\mathbf{M}(X)$.

By a complex Borel measure on X, we will always be considering *finite* measures: given a complex Borel measure μ, the series $\sum_{i=1}^{\infty} \mu(U_i)$ converges absolutely for any sequence $\{U_i\}_{i=1}^{\infty}$ of mutually disjoint Borel subsets of X. This is formalized in the next definition.

Definition 2. A *complex measure* on the Borel σ-algebra $\mathfrak{B}(X)$ is a set function μ satisfying

(i) $\mu(\varnothing) = 0$;

(ii) given any sequence of mutually disjoint subsets $\{U_i\}_{i=1}^{\infty} \subset \mathfrak{B}(X)$, the series $\sum_{i=1}^{\infty} \mu(U_i)$ converges absolutely to $\mu(\bigcup_{i=1}^{\infty} U_i)$.

We will denote the collection of all complex Borel measures on X as $\mathbf{M}(X)$. For $\mu, \nu \in \mathbf{M}(X)$ and $\alpha \in \mathbb{C}$, we define $\alpha\mu$ and $\mu + \nu$ by

$$(\alpha\mu)(U) = \alpha\mu(U);$$
$$(\mu + \nu)(U) = \mu(U) + \nu(U),$$

for all $U \in \mathfrak{B}(X)$. Under this definition, $\alpha\mu$ and $\mu+\nu$ are also complex Borel measures on X, and $\mathbf{M}(X)$ is a complex linear space.

Definition 3. Given a Borel set $U \in \mathfrak{B}(X)$, a *finite Borel partition* P of U is a finite collection of Borel sets $P \subset \mathfrak{B}(X)$ that are mutually disjoint. The *total variation* of a complex measure μ is the finite, real-valued Borel measure defined by

$$|\mu|(U) = \sup \left\{ \sum_{U \in P} |\mu(U_i)| : P \text{ is a finite Borel partition of } U \right\} \tag{3}$$

for all $U \in \mathfrak{B}(X)$.

In the above, the choice of using finite partitions is somewhat arbitrary; infinite partitions would construct the same measure. Also, for our purposes, we will not be needing any other types of partitions, so we will refer to finite Borel partitions simply as "partitions". It is *definitely* not clear from the definition above that $|\mu|$ is a measure, let alone finite. Convincing us of this is the burden of the next theorem.

Theorem 4. The positive set function μ defined by Equation (3) is a finite Borel measure on X.

Proof. It is clear from the definition that $|\mu|(\varnothing) = 0$. Moreover, let $\{U_i\}_{i=1}^{\infty} \subset \mathfrak{B}(X)$ be a sequence of mutually disjoint sets and define $U = \bigcup_{i=1}^{\infty} U_i$. Then given a partition P of U, each collection $\{V \cap U_j\}_{V \in P}$ is a partition of U_j and therefore

$$\sum_{V \in P} |\mu(V)| = \sum_{V \in P}^{n} \left| \sum_{j=1}^{\infty} \mu(U_j \cap V) \right| \leqslant \sum_{j=1}^{\infty} \sum_{V \in P} |\mu(U_j \cap V)| \leqslant \sum_{j=1}^{\infty} |\mu|(U_j).$$

Since the partition of U was chosen arbitrarily, we may conclude

$$|\mu|(U) \leqslant \sum_{j=1}^{\infty} |\mu|(U_j).$$

To prove the converse inequality, let $\epsilon > 0$. Then for each integer $j \geqslant 1$, there exists a partition P_j of U_j such that

$$|\mu|(U_j) \leqslant \sum_{V \in P_j} |\mu(V)| + \epsilon 2^{-j}.$$

Now for any integer $k \geqslant 1$, define $V_k = \bigcup_{j=k}^{\infty} U_j$. We have that for all integers $k \geqslant 1$, the collection $\{V : 1 \leqslant j \leqslant k, V \in P_j\} \cup \{V_{k+1}\}$ forms a partition of U so that

$$\sum_{j=1}^{k} |\mu|(U_j) \leqslant \sum_{j=1}^{k} \left(\sum_{V \in P_j} |\mu(V)| + \epsilon 2^{-j} \right) \leqslant |\mu(V_{k+1})| + \sum_{j=1}^{k} \left(\sum_{V \in P_j} |\mu(V)| + \epsilon 2^{-j} \right)$$
$$\leqslant |\mu|(U) + \epsilon \sum_{j=1}^{k} 2^{-j} < |\mu|(U) + \epsilon.$$

Taking the limit as $k \to \infty$ and then as $\epsilon \to 0$ provides us with the reverse inequality, and we conclude

$$|\mu|(U) = \sum_{j=1}^{\infty} |\mu|(U_j).$$

Hence $|\mu|$ is a measure. If $|\mu|$ is not finite, then $|\mu|(U) = \infty$ for some $U \in \mathfrak{B}(X)$. Thus for all integers $j \geqslant 1$, there exists a partition P_j of U such that $\sum_{V \in P_j} |\mu(V)| > j$. Taking the limit as $j \to \infty$, we obtain a countable partition P of U such that $\sum_{V \in P} |\mu(V)| = \infty$. This contradicts the assumption that μ is a complex measure. □

We use the total variation $|\mu|$ of a complex measure μ to determine which functions are integrable with respect to μ. In particular, we have the following definition.

Definition 5. If ν is a positive measure, then a ν-measureable function f is said to be *ν-integrable* if $\int |f| d\nu < \infty$, and we write $f \in L_1(\nu)$. If μ is a complex measure, f is said to be *μ-integrable* if $f \in L_1(|\mu|)$, and we write $f \in L_1(\mu)$. That is, $L_1(\mu) = L_1(|\mu|)$ by definition.

If $f \in L_1(|\mu|)$ and ν is the measure defined by $\nu(U) = \int_U f d\mu$, we use the usual notation $d\nu = f d\mu$. Before exploring the properties of the total variation, it is useful to recall an important result from measure theory.

Definition 6. Let μ and ν be measures on the same measure space X. Then μ is *absolutely continuous* with respect to ν, denoted ($\mu \ll \nu$) if $\mu(U) = 0$ for all sets $U \subset X$ of ν-measure 0.

Definition 7. Let μ and ν be measures on the same measure space X. Then μ and ν are said to be *mutually singular*, denoted $\mu \perp \nu$, if there exist measureable sets $U_\mu, U_\nu \subset X$ satisfying $U_\mu \cap U_\nu = \varnothing$, $U_\mu \cup U_\nu = X$, $\mu(U) = 0$ for all measureable $U \subset U_\nu$, and $\nu(U) = 0$ for all measureable $U \subset U_\mu$. That is, the measures μ and ν 'live on different sets'.

Theorem 8 (Lebesgue-Radon-Nikodym Theorem)**.** If μ is a complex measure and ν is a σ-finite positive measure on X, there exists a complex measure λ and a function $f \in L_1(\nu)$ such that λ and μ are mutually singular and $d\nu = d\lambda + f d\mu$. Moreover, the function f and the measure λ are unique up to sets of μ-measure 0.

Proof. See, for example, [3] pp. 90-93. □

In the case that $\mu \ll \nu$, the above theorem guarantees the existence of a measure $f \in L_1(\nu)$ such that $d\mu = f d\nu$, and we use the notation $f = d\mu/d\nu$. It is clear from the definition of $|\mu|$ that $\mu \ll |\mu|$, so as a particular case, we are always guaranteed the existence of $d\mu/d|\mu| \in L_1(|\mu|)$. Using this tool, we are prepared for an investigation of the properties of $|\mu|$. The first main property we establish is the existence of a norm. To do so, we first establish some preliminary results.

Lemma 9. Let μ be a complex measure and ν a σ-finite positive measure with $f \in L_1(\nu)$ satisfying $d\mu = f d\nu$. Then $d|\mu| = |f| d\nu$.

Proof. Let μ, ν and f satisfy the assumptions of the theorem, and let $U \in \mathfrak{B}(X)$. Given any partition P of U, we have that

$$\sum_{V \in P} |\mu(V)| = \sum_{V \in P} \left| \int_V f d\nu \right| \leqslant \sum_{V \in P} \int_V |f| d\nu = \int_U |f| d\nu.$$

Since the choice of partition was arbitrary, we may conclude that $|\mu|(U) \leqslant \int_U |f| d\nu$.

To prove the converse, suppose that U is compact. Then f may be approximated on U by simple functions[2]. In particular, for all $\epsilon > 0$, there exists a partition P of U and a collection of complex numbers $\{\alpha_V : V \in P\}$ such that $|f - \sum_{V \in P} \alpha_V 1_V| < \epsilon/2\nu(U)$ on U. It follows that

$$\begin{aligned}\int_U |f| d\nu &\leqslant \sum_{V \in P} |\alpha_V| \nu(V) + \epsilon/2 = \left|\sum_{V \in P} \alpha_V \nu(V)\right| + \epsilon/2 \\ &\leqslant \left|\int_U f d\nu\right| + \epsilon = |\mu(U)| + \epsilon < |\mu|(U) + \epsilon.\end{aligned}$$

Taking the limit as $\epsilon \to 0$, we find that $\int_U |f| d\nu \leqslant |\mu|(U)$. Combining this with our previous inequality proves $d|\mu| = |f| d\nu$ for all compact sets. Since both $|\mu|$ and ν are σ-finite, this proves our lemma. □

Corollary 10. Let $\mu \in \mathbf{M}(X)$. Then $\left|\frac{d\mu}{d|\mu|}(x)\right| = 1$ for $|\mu|$-a.e. $x \in X$.

Proof. As a direct result of the definition of total variation, the total variation of a finite positive measure is the measure itself. Since $d\mu = \frac{d\mu}{d|\mu|} d|\mu|$, Lemma (9) tells us

$$d|\mu| = \left|\frac{d\mu}{d|\mu|}\right| d|\mu|.$$

The statement of the corollary follows immediately. □

Corollary 11. Let $\mu, \nu \in \mathbf{M}(X)$, and $f \in L_1(|\nu|)$ with $d\mu = f d\nu$. Then $d|\mu| = |f| d|\nu|$.

Proof. Note that $d\mu = f \frac{d\nu}{d|\nu|} d|\nu|$. Lemma (9) and the previous corollary tell us

$$d|\mu| = \left|f \frac{d\nu}{d|\nu|}\right| d|\nu| = |f| d|\nu|.$$

[2] This is a standard fact when f is real and positive. If f is complex, it can be accomplished by decomposing f into a linear combination of positive, real-valued functions.

□

Corollary 12. Let $\mu \in \mathbf{M}(G)$ and $f \in L_1(|\mu|)$. Then

$$\left|\int_X f d\mu\right| \leqslant \int |f| d|\mu|.$$

Proof. Let $d\nu = f d\mu$. Then $\nu \in \mathbf{M}(X)$ and $d|\nu| = |f|d|\mu|$. It follows that

$$\left|\int_X f d\mu\right| = |\nu(X)| \leqslant |\nu|(X) = \int_X |f| d|\mu|.$$

□

Theorem 13. Let $\mu_1, \mu_2 \in \mathbf{M}(X)$. Then $|\mu_1 + \mu_2| \leqslant |\mu_1| + |\mu_2|$ and $\mathbf{M}(X)$ is a normed linear space with norm defined by

$$\|\mu\| = |\mu|(X). \tag{4}$$

Proof. Define $\nu = |\mu_1| + |\mu_2|$. Then $\mu_i \ll \nu$ and there exist functions $f_i \in L_1(\nu)$ such that $d\mu_i = f_i d\nu$ for $i = 1, 2$. By the previous lemma, $d|\mu_i| = |f_i| d\nu$. Thus for any $U \in \mathfrak{B}(X)$, we have that

$$|\mu_1(U) + \mu_2(U)| = \left|\int_U f_1 + f_2 d\nu\right| \leqslant \int_U |f_1| d\nu + \int_U |f_2| d\nu = |\mu_1|(U) + |\mu_2|(U).$$

In particular, this shows that $\|\mu_1 + \mu_2\| = |\mu_1 + \mu_2|(X) \leqslant |\mu_1|(X) + |\mu_2|(X) = \|\mu_1\| + \|\mu_2\|$. Moreover, $d\mu_1 = f_1 d\nu$, so that $d(\alpha\mu_1) = \alpha d\mu_1 = \alpha f_1 d\nu$. It follows from the previous lemma that $d|\alpha\mu_1| = |\alpha||f_1| d\nu$, and therefore $\|\alpha\mu_1\| = |\alpha|\|\mu_1\|$. Since the choice of $\mu_1, \mu_2 \in \mathbf{M}(X)$ and $\alpha \in \mathbb{C}$ was arbitrary, all that is left to show is that $\|\mu\| \geqslant 0$, with equality if and only if μ is the zero measure.

If $\mu \in M(X)$ is the zero measure, it follows from the definition that $\|\mu\| = 0$.

If $\mu \neq 0$, then there exists a set $U \in \mathfrak{B}(X)$ such that $\mu(U) \neq 0$. Therefore $\|\mu\| = |\mu|(X) \geqslant |\mu(U)| + |\mu(X \backslash U)| \geqslant |\mu(U)| > 0$. This proves our theorem. $\square$

The total variation of a measure μ may also be expressed in terms of functions in $C_0(X)$. This is a very important fact, as it will be the key in relating the norm on $\mathbf{M}(X)$ to the norm on $C_0^*(X)$.

Theorem 14. Let $\mu \in \mathbf{M}(X)$. Then for all $U \in \mathfrak{B}(X)$ we have that

$$|\mu|(U) = \sup\left\{\left|\int_U f d\mu\right| : f \in C_0(X),\ |f| \leqslant 1\right\}. \tag{5}$$

More generally, given any real, positive function $f \in L_1(|\mu|)$, we have that

$$\int_X f d|\mu| = \sup\left\{\left|\int_X g d\mu\right| : g \in C_0(X),\ |g| \leqslant f\right\}. \tag{6}$$

Proof. Let $U \in \mathfrak{B}(X)$ and define $c = \sup\left\{\left|\int_U f d\mu\right| : f \in C_0(X),\ |f| \leqslant 1\right\}$. Let $\epsilon > 0$ and fix a partition P of U. For every $V \in P$, there exists a compact set $V_\epsilon \subset V$ such that $\mu(V \triangle V_\epsilon) < \epsilon/n$, where n is the number of members of P. Define $\varphi = \sum_{i=1}^n \operatorname{sign}(\mu(V_\epsilon))1_{V_\epsilon} \in C_0(X)$, and let $f \in C_0(X)$ satisfy $|f - \varphi| < \epsilon/|\mu|(U)$ and $|f| \leqslant 1$. Then

$$\sum_{V \in P} |\mu(V)| - \epsilon < \sum_{V \in P} |\mu(V_\epsilon)| = \int_U \varphi d\mu \leqslant \int_U |f - \varphi| d|\mu| + \left|\int_U f d\mu\right| \leqslant c + \epsilon.$$

Taking the limit as $\epsilon \to 0$, and noting that the partition P was arbitrary, we conclude $|\mu|(V) \leqslant c$.

Conversely, let $f \in C_0(X)$ with $|f| \leqslant 1$. Then $|\int_U f d\mu| \leqslant \int_U |f| d|\mu| \leqslant |\mu|(U)$. This proves the identity in Equation (5).

Let $f \in C_{00}(X)$ be a real, positive function and define a complex measure ν by

$d\nu = f d|\mu|$. Suppose that $h \in C_0(X)$ with $|h| \leqslant f$. Then define a function g by

$$g(x) = \begin{cases} h(x)/f(x), & \text{for } f(x) \neq 0; \\ 0, & \text{for } f(x) = 0. \end{cases}$$

Then $g \in C_{00}(X)$, $|g| \leqslant 1$ and $fg = h$ for $|\mu|$-a.e. $x \in X$. Conversely, if $g \in C_0(X)$ with $|g| \leqslant 1$, then $fg \in C_{00}(X)$ and $|fg| \leqslant f$. It follows that

$$\begin{aligned} \int_X f d|\mu| &= \sup \left\{ \left| \int_X g d\nu \right| : g \in C_0(X),\ |g| \leqslant 1 \right\} \\ &= \sup \left\{ \left| \int_X h d\mu \right| : h \in C_0(X),\ |h| \leqslant f \right\}. \end{aligned}$$

This proves the identity in Equation (6) for real, positive $f \in C_{00}(X)$. The identity for all real, positive $f \in L_1(\mu)$ follows immediately from the monotone convergence theorem by approximating f with functions in C_{00}. $\square$

1.3. An Isometry Between $\mathbf{M}(X)$ and $C_0(X)^*$

For every $\mu \in \mathbf{M}(X)$, there exists a linear functional $T_\mu \in C_0(X)^*$ satisfying

$$\langle f, T_\mu \rangle = \int_X f d\mu. \tag{7}$$

The primary content of this section is that every linear functional in $C_0(X)^*$ can be described in this way, and in fact that the correspondence $\Phi : \mathbf{M}(X) \to C_0(X)^*$ defined by $\Phi : \mu \mapsto T_\mu$ is an isometry.

We begin by proving that the mapping Φ is a well-defined norm-preserving monomorphism of Φ onto $\mathbf{M}(X)$.

Theorem 15. Let $\mu \in \mathbf{M}(X)$. Then T_μ defined by Equation (7) is a bounded linear functional in $C_0(X)$ and $\|T_\mu\| = \|\mu\|$. Moreover, the mapping $\Phi : \mu \mapsto T_\mu$ is a norm-preserving monomorphism of $\mathbf{M}(X)$ into $C_0^*(X)$.

Proof. From the properties of integration, T_μ is a linear functional. From Equation (5)

$$\begin{aligned}\|\mu\| &= \sup\left\{\left|\int_X f d\mu\right| : f \in C_0(X),\ |f| \leqslant 1\right\} \\ &= \{|\langle f, T_\mu\rangle| : f \in C_0(X),\ |f| \leqslant 1\} = \|T_\mu\|,\end{aligned}$$

and therefore Φ preserves norms. If $\mu, \nu \in \mathbf{M}(X)$ and $\alpha \in \mathbb{C}$, then

$$\langle f, T_{\mu+\nu}\rangle = \int_X f d(\mu+\nu) = \int_X f d\mu + \int_X f d\nu = \langle f, T_\mu\rangle + \langle f, T_\nu\rangle,$$

for all $f \in C_0(X)$. Thus $T_{\mu+\nu} = T_\mu + T_\nu$, and similarly $T_{\alpha\mu} = \alpha T_\mu$. Thus Φ is a homomorphism. To show that it is a monomorphism, note that if $T_\mu = 0$, then for any $U \in \mathfrak{B}(X)$, $\langle 1_U, T_\mu\rangle = \mu(U) = 0$. It follows that $\mu = 0$. This proves our theorem. □

To show that Φ is an isometry, all that is left to show is that Φ is onto. Equivalently, this is saying that every linear functional $T \in C_0^*(X)$ is of the form Equation (7) with $T = T_\mu$ for some measure $\mu \in \mathbf{M}(X)$. This is a standard fact in the case that T is a real, positive linear functional. The general case requires some machinery that will be used to express any linear functional $T \in C_0^*(X)$ as a linear combination of real, positive linear functionals. The key to this is the "linear modulus" of a linear operator.

In order to define the linear modulus, we first note that if B is a Banach space of complex-valued functions such that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are in B for every $f \in B$, then the value of a linear functional T on B is completely determined by its value on positive, real-valued functions. In particular,

$$\begin{aligned}T(f) &= T(\max(\mathrm{Re}(f),0)) - T(-\min(\mathrm{Re}(f),0)) \\ &\quad + i(T(\max(\mathrm{Im}(f),0)) - T(-\min(\mathrm{Im}(f),0)))\end{aligned}$$

for every $f \in B$. Thus, if we define a linear functional T on the subset of all real-valued functions in B it extends uniquely to a linear functional on B.

Definition 16. Let T be a bounded linear functional on a Banach space B of complex-valued functions such that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are in B for every $f \in B$. The linear modulus $|T|$ of T is the functional on B defined by

$$|T|(f) = \sup\{|T(g)| : g \in B,\ |g| \leqslant |f|\} \tag{8}$$

for all positive, real-valued functions $f \in B$.

This definition looks *very* similar to the definition of the total variation of a signed measure, and for good reason. In fact, we will be able to show that $\Phi : |\mu| \mapsto |T_\mu|$. However, before we do that, we should convince ourselves that $|T|$ is in fact a bounded linear functional on B. This is the task of the next theorem.

Theorem 17. Let B be a Banach space of complex-valued functions such that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are in B for every $f \in B$, and let T be a bounded linear functional on B. Then $|T|$ as defined by Equation (8) is a bounded, positive linear functional on B with $|T|(f) \leqslant |T|(g)$ for real, positive functions f, g with $f \leqslant g$ and $\|T\| = \||T|\|$.

Proof. Suppose that $f_1, f_2 \in B$ are real-valued, positive functions. Then for any $h \in B$ with $|h| \leqslant f_1 + f_2$ we have that the functions h_1 and h_2 defined by $h_2 =$

$\min(|h|, f_2)\text{sign}(h)$ and $h_1 = (|h| - h_2)\text{sign}(h)$ satisfy $|h_i| < f_i$ and $h_1 + h_2 = h$. Thus

$$|T(h)| = |T(h_1 + h_2)| \leqslant |T(h_1)| + |T(h_2)| \leqslant |T|(f_1) + |T|(f_2).$$

Since this is true for any $h \in B$ with $|h| \leqslant f_1 + f_2$, we may conclude that $|T|(f_1 + f_2) \leqslant |T|(f_1) + |T|(f_2)$. Moreover, for all $\epsilon > 0$, there exists $h_i \in B$ such that $|h_i| \leqslant f_i$ and $|T|(f_i) \leqslant |T(h_i)| + \epsilon/2$, for $i = 1, 2$. Define $\lambda_i \in \mathbb{T}$ such that $|T(h_i)| = \lambda_i T(h_i)$. Then $|\lambda_1 h_1 + \lambda_2 h_2| \leqslant |h_1| + |h_2| \leqslant f_1 + f_2$ and

$$\begin{aligned} |T|(f_1) + |T|(f_2) - \epsilon < |T(h_1)| + |T(h_2)| &= ||T(h_1)| + |T(h_2)|| \\ &= |\lambda_1 T(h_1) + \lambda_2 T(h_2)| \leqslant |T|(f_1 + f_2). \end{aligned}$$

Thus $|T|(f_1) + |T|(f_2) \leqslant |T|(f_1 + f_2)$. Combining this with the previous inequality, we conclude that $|T|(f_1) + |T|(f_2) = |T|(f_1 + f_2)$. Moreover, for any real number $\alpha > 0$,

$$\begin{aligned} |T|(\alpha f_1) &= \sup\{|T(h_1)| : |h_1| \leqslant \alpha f_1\} = \sup\{|T(\alpha h_1)| : |h_1| \leqslant f_1\} \\ &= \sup\{\alpha|T(h_1)| : |h_1| \leqslant f_1\} = \alpha|T|(f_1). \end{aligned}$$

Hence $|T|$ is a positive linear functional on the set of all real, positive functions in B. It follows that T is a positive linear functional on B. Moreover, the definition of T tells us that $T(f_1) \leqslant T(f_2)$ for $f_1 \leqslant f_2$.

The last thing that we must show is that $\|T\| = \||T|\|$. We note that

$$\begin{aligned} \||T|\| &= \sup\{||T|(f)| : f \in B,\ |f| \leqslant 1\} \\ &= \sup\{|\sup\{|T(g)| : g \in B,\ |g| \leqslant f\}| : f \in B,\ |f| \leqslant 1\} \\ &= |\sup\{|T(g)| : g \in B,\ |g| \leqslant 1\}| = |(\|T\|)| = \|T\|. \end{aligned}$$

This proves our theorem. □

Using the linear modulus, we may construct "maximum" and "minimum" linear functionals from two arbitrary linear functionals. The notion of and notation for this are both provided in the next definition.

Definition 18. Let T_1, T_2 be linear functionals on a Banach space B of complex-valued functions such that $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are in B for every $f \in B$. We define the *minimum* $\min(T_1, T_2)$ and the *maximum* $\max(T_1, T_2)$ of two linear operators T_1 and T_2, by

$$\max(T_1, T_2)(f) = \frac{1}{2}(T_1(f) + T_2(f) + |T_1 + T_2|(f)) \tag{9}$$

$$\min(T_1, T_2)(f) = \frac{1}{2}(T_1(f) + T_2(f) - |T_1 + T_2|(f)). \tag{10}$$

It is clear from the definition $\max(T_1, T_2)$ and $\min(T_1, T_2)$ are linear functionals. In addition, we call two linear functionals T_1 and T_2 *mutually singular* if $\min(T_1, T_2) = 0$.

The maximum and minimum defined above provides a partial ordering on the collection of all linear functionals on B. In particular, we say that $T_1 \leqslant T_2$ if $\min(T_1, T_2) = T_1$ or equivalently $\max(T_1, T_2) = T_2$. The next theorem outlines some important properties of these linear functionals.

Theorem 19. Let T_1, T_2 be linear functionals on a Banach space B of complex-valued functions. Then $\max(T_1, T_2) = \max(T_2, T_1)$, $\min(T_1, T_2) = \min(T_2, T_1)$, $\max(T_1, T_2) = \max(T_1 - T_2, 0) + T_2$, and $\min(T_1, T_2) = \min(T_1 - T_2, 0) + T_2$. Also $\min(T_1, T_2) = T_1 - \max(T_1 - T_2, 0)$.

Proof. The properties $\max(T_1, T_2) = \max(T_2, T_1)$ and $\min(T_1, T_2) = \min(T_2, T_1)$

follow immediately from the definitions. Moreover

$$\begin{aligned}\min(T_1 - T_2, 0) + T_2 &= \frac{1}{2}(T_1 - T_2 + 0 - |T_1 - T_2 + 0|) + T_2 \\ &= \frac{1}{2}(T_1 + T_2 - |T_1 - T_2|) = \min(T_1, T_2).\end{aligned}$$

The proof of $\max(T_1, T_2) = \max(T_1 - T_2, 0) + T_2$ is similar. Lastly, we have that

$$T_1 - \max(T_1 - T_2, 0) = T_1 - \max(T_1, T_2) + T_2 = (T_1 + T_2 - |T_1 - T_2|) = \min(T_1, T_2).$$

□

The machinery just defined may be used to rip apart a linear functional T into a linear combination of positive linear functionals that is unique in a certain sense.

Theorem 20. Let $T \in C_0^*(X)$. Then T may be decomposed into a direct sum $T = T_1 - T_2 + i(T_3 - T_4)$, where T_i is a positive bounded linear functional for $1 \leqslant i \leqslant 4$ and the decomposition is unique in the sense that T_1 and T_2 are mutually singular and T_3 and T_4 are mutually singular. If $T = T_1' - T_2' + i(T_3' - T_4')$ for some positive bounded linear functionals T_i' and T_1' and T_2' are mutually singular, and T_3 and T_4 are mutually singular, then $T_i = T_i'$ for $1 \leqslant i \leqslant 4$.

Proof. Define linear functionals T_R and T_I by $T_R(f) = \mathrm{Re}(T(f))$ and $T_I(f) = \mathrm{Im}(T(f))$ for all real-valued, positive $f \in C_0(X)$. Since T is bounded, so too are T_R and T_I. Define $T_1 = \max(T_R, 0)$, $T_2 = T_1 - T_R$, $T_3 = \max(T_I, 0)$ and $T_4 = T_3 - T_I$. Then T_i is a bounded linear functional for $1 \leqslant i \leqslant 4$, $T_1 - T_2 = T_R$, and $T_3 - T_4 = T_I$, so that $T = T_1 - T_2 + i(T_3 - T_4)$. Also, if $T_1(f) = 0$, then $0 > T_R(f) = -T_2(f)$, and therefore $\min(T_1, T_2)(f) = T_1(f) = 0$. If $T_1(f) \neq 0$, then $T_1(f) = \max(T_R, 0)(f) = T_R(f) > 0$, so that $T_2(f) = 0$ and $\min(T_1, T_2)(f) = T_2(f) = 0$. Thus $\min(T_1, T_2) = 0$. Similarly $\min(T_3, T_4) = 0$.

If T_i' satisfy the assumptions in the theorem above, then $0 = \min(T_1', T_2') = T_1' - \max(T_1' - T_2', 0)$, and therefore $T_1' = \max(T_1' - T_2', 0) = \max(T_R, 0)$. It follows that $T_1 = T_1'$ and therefore $T_2 = T_2'$. Similarly $T_3 = T_3'$ and $T_4 = T_4'$. This proves our theorem. □

At this point, we have set up all the machinery required to produce a measure from $T \in C_0^*$. It is a standard fact in analysis[3] that for any positive, linear functional I on C_{00}^+, there exists a measure ι defined on a σ-algebra of subsets of X containing the Borel σ-algebra such that for $f \geqslant 0$

$$\int_X f d\iota = \begin{cases} \langle f, I \rangle, & f \in C_{00}^+; \\ \sup \langle g, I \rangle : g \in C_{00}^+, g \leqslant f & f \in \mathfrak{M}^+; \\ \inf \langle g, I \rangle : g \in \mathfrak{M}^+, f \leqslant g & \text{otherwise.} \end{cases} \tag{11}$$

Since C_{00} is a dense linear subspace of C_0, the values of bounded linear functionals on C_0 are completely determined by their values on C_{00}, so any bounded linear functional on C_{00} can be extended uniquely to a bounded linear functional on C_0. Thus in particular, the measure ι corresponding to I satisfies the equation

$$\langle f, I \rangle = \int_X f d\iota$$

for all $f \in C_0$. We can then use the decomposition of a bounded linear functional into bounded positive linear functionals described above to construct a measure for a more general linear functional.

Theorem 21. Let $T \in C_0^*$. Then there exists a measure $\mu \in \mathbf{M}(X)$ such that T and μ satisfy Equation (7) with $T_\mu = T$. In particular, Φ is surjective. Moreover,

[3]This is a standard application of the Riesz Representation Theorem. See, for example, [3] pp. 212.

$\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$ with μ_i a finite, positive measure and $\Phi(\mu_i) = T_i$ for $1 \leqslant i \leqslant 4$, where the T_i are the uniquely defined bounded linear functionals from Theorem (20).

Proof. Let T_i with $1 \leqslant i \leqslant 4$ be the uniquely defined positive, bounded linear functionals from Theorem (20). Then for each T_i, there exists a positive measure μ_i such that $\langle f, T_i \rangle = \int_X f d\mu_i$. Since T_i is bounded, μ_i must be finite. Set $\mu = \mu_1 - \mu_2 + i(\mu_3 - \mu_4)$. It follows that

$$\begin{aligned}\int_X f d\mu &= \int_X f d\mu_1 - \int_X f d\mu_2 + i\left(\int_X f d\mu_3 - \int_X f d\mu_4\right) \\ &= \langle f, T_1 \rangle - \langle f, T_2 \rangle + i\left(\langle f, T_3 \rangle - \langle f, T_4 \rangle\right) = \langle f, T \rangle.\end{aligned}$$

This proves our theorem. □

1.4. Decomposition Relationships

This section is devoted to a couple of results which are interesting but inessential to the further development of the theory. The main reason we bring them up at this point is to provide a better idea what our decomposition above actually did, at least in terms of measures and the usual decompositions of measures. We will show in particular that if $\Phi : \mu \mapsto T_\mu$, then $|\mu| \mapsto |T_\mu|$ and we will relate the decomposition of T in Theorem (20) to the Jordan decomposition of a signed measure. The first result has already been obtained, but has yet to be formally mentioned.

Theorem 22. Let $\mu \in \mathbf{M}(X)$ and $T \in C_0^*(X)$ with $\Phi : \mu \mapsto T$. Then $\Phi : |\mu| \mapsto |T|$.

Proof. This is simply a restatement of Equation (6). □

The second result will require two results from measure theory.

Theorem 23 (Hahn Decomposition)**.** Let μ be a signed measure on a σ-algebra Ω of subsets of X. A positive subset of X is a μ-measureable set E such that $\mu(F) \geqslant 0$

for all $F \subset E$. A negative subset of X is a μ-measureable set E such that $\mu(F) \leqslant 0$ for all $F \subset E$. Then there exists a positive set X_+ and negative set X_- such that $X_+ \cup X_- = X$, $X_+ \cap X_- = 0$, $\mu(X_+) > 0$, and $\mu(X_-) < 0$. The choices of X_- and X_+ are unique up to sets of measure 0.

Proof. See, for example, [3] pp. 86-87. □

Theorem 24. (Jordan Decomposition) For a signed measure μ, there exist unique positive measures μ^+ and μ^- such that $\mu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$.

Proof. Let $X = X_+ \cup X_-$ be a Hahn decomposition for μ. Define $\mu^+(E) = \mu(E \cap X_+)$ and $\mu^-(E) = -\mu(E \cap X_-)$ for any μ-measureable set E. Clearly μ^+ and μ^- are positive measures on the same measure space as μ and and satisfy $\mu = \mu^+ - \mu^-$ and $\mu^+ \perp \mu^-$. Let ν^+ and ν^- be two positive measures on the measure space of μ such that $\mu = \nu^+ - \nu^-$ and E and F are two μ-measureable sets satisfying $E \cap F = \varnothing$, $E \cup F = X$, and $\nu^-(E) = \nu^+(F) = 0$. Then E and F is another Hahn decomposition for μ and therefore $\nu^+(A) = \nu^+(A \cap E) = \mu(A \cap E) = \mu(A \cap X_+) = \mu^+(A)$. Thus $\nu^+ = \mu^+$. Similarly $\nu^- = \mu^-$. □

For signed measures, the total variation is usually defined in terms of the Jordan decomposition. In particular, $|\mu|$ is defined to be $\mu^+ + \mu^-$. This is necessary, since not all signed measures are finite measures and our definition of the total variation applies only to finite measures. In the case of finite signed measures, this definition turns out to be no different than ours, as the next theorem shows.

Theorem 25. Let μ be a finite signed measure. Then $|\mu| = \mu^+ + \mu^-$, where μ^+ and μ^- are the components of the Jordan decomposition of μ.

Proof. Let X_+ and X_- be the positive and negative sets, respectively, of the Hahn decomposition of X. Let $U \in \mathfrak{B}(X)$. Then for any partition P of U,

$$\sum_{V\in P} |\mu(V)| = \sum_{V\in P} |\mu(X_+ \cap V) + \mu(X_- \cap V)| \leqslant \sum_{V\in P} (|\mu(X_+ \cap V)| + |\mu(X_- \cap V)|)$$
$$= \sum_{V\in P} (\mu^+(V) + \mu^-(V)) = \mu^+(U) + \mu^-(U).$$

Since the partition was arbitrary, we may conclude that $|\mu|(V) \leqslant \mu^+(V) + \mu^-(V)$. Conversely, $\{X_+ \cap U, X_- \cap U\}$ is a partition of U and therefore

$$\mu^+(U) + \mu^-(U) = |\mu(X_+ \cap U)| + |\mu(X_- \cap U)| \leqslant |\mu|(U).$$

Combined with the previous inequality, this proves $\mu^+ + \mu^- = |\mu|$. □

Our next task will be to decompose a measure $\mu \in \mathbf{M}(X)$ into real and imaginary components.

Theorem 26. Let $\mu \in \mathbf{M}(X)$. For all Borel-measureable subsets $U \in \mathfrak{B}(X)$, we define

$$\mu_R(U) = \mathrm{Re}(\mu(U)); \tag{12}$$

$$\mu_I(U) = \mathrm{Im}(\mu(U)). \tag{13}$$

The real-valued set functions μ_R and μ_I are finite Borel measures on X (and therefore in $\mathbf{M}(X)$).

Proof. We will prove the theorem for μ_R, since the proof for μ_I is similar. Let $\mu \in \mathbf{M}(X)$. Since μ is a finite measure, it is clear that $\mu_R(U) \leqslant |\mu(U)| < \infty$ for all $U \in \mathfrak{B}(X)$, and therefore μ_R is finite. Moreover, $\mu_R(\varnothing) = \mathrm{Re}(\mu(\varnothing)) = 0$ and for any sequence of mutually disjoint sets $\{U_i\}_{i=1}^{\infty} \subset \mathfrak{B}(X)$, we have that

$$\mu_R\left(\bigcup_{i=1}^{\infty} U_i\right) = \mathrm{Re}\left(\mu\left(\bigcup_{i=1}^{\infty} U_i\right)\right) = \mathrm{Re}\left(\sum_{i=1}^{\infty} \mu(U_i)\right) = \sum_{i=1}^{\infty} \mathrm{Re}(\mu(U_i)) = \sum_{i=1}^{\infty} \mu_R(U_i).$$

This proves that μ_R is a measure. $\square$

It follows from this theorem that any complex-valued measure μ may be expressed as a linear combination $\mu = \mu_R + i\mu_I$ of it's real and imaginary components. Combining this with the Jordan decompositions of μ_R and μ_I, we may express μ as a sum of positive measures $\mu = \mu_R^+ - \mu_R^- + i(\mu_I^+ - \mu_I^-)$. Moreover, this decomposition is unique in the sense that the Jordan decompositions of μ_R and μ_I are unique. The next theorem shows that this decomposition is directly related to the decomposition of $T \in C_0^*(X)$ given by Theorem (20).

Theorem 27. Let $T \in C_0^*(X)$ and $\mu \in \mathbf{M}(X)$ with $\Phi : \mu \mapsto T$. Let $T = T_1 - T_2 + i(T_3 - T_4)$ be the unique decomposition of T given by Theorem (20). Then $T_1 = \Phi(\mu_R^+)$, $T_2 = \Phi(\mu_R^-)$, $T_3 = \Phi(\mu_I^+)$ and $T_4 = \Phi(\mu_I^-)$.

Proof. Define T_R and T_I by $T_R(f) = \mathrm{Re}(T(f))$ and $T_I = \mathrm{Im}(T(f))$ for all real, positive $f \in C_0(X)$. Then, as before, T_R and T_I are real-valued, bounded linear functionals on $C_0(X)$. Moreover, it is easy to see that $T_R = \mu_R$ and $T_I = \mu_I$. Moreover, $|\mu_R| = \mu_R^+ + \mu_R^-$, so that $|T_R| = \Phi(\mu_R^+ + \mu_R^-)$ and

$$T_1 = \max(T_R, 0) = \frac{1}{2}(T_R + |T_R|) = \frac{1}{2}(\Phi(\mu_R^+ - \mu_R^-) + \Phi(\mu_R^+ + \mu_R^-)) = \Phi(\mu_R^+).$$

It follows that $T_2 = T_1 - T_R = \Phi(\mu_R^+) - \Phi(\mu_R^+ - \mu_R^-) = \Phi(\mu_R^-)$. The proof that $T_3 = \Phi(\mu_I^+)$ and $T_4 = \Phi(\mu_I^-)$ is similar. $\square$

It is important to note that it is *not true* that $|\mu| = \mu_R^+ + \mu_R + \mu_I^+ + \mu_I^-$ unless $\mu_R \perp \mu_I$. This is simply a consequence of the fact that in general $|T_1+T_2| \neq |T_1|+|T_2|$

for $T_1, T_2 \in C_0^*(X)$.

Before ending this section, we consider one last result which asserts that the isomorphism Φ is order-preserving.

Theorem 28. Let $\mu_i \in \mathbf{M}(X)$ be positive measures and let $T_i \in C_0(X)^*$ with $\mu_i = \Phi(T_i)$ for $i = 1, 2$. Then $\mu_1 \leqslant \mu_2$ if and only if $T_1 \leqslant T_2$.

Proof. If $\mu_1 \leqslant \mu_2$, then for any real-valued, positive function $f \in C_0(X)$, we have that

$$T_1(f) = \int_X f d\mu_1 \leqslant \int_X f d\mu_2 = T_2(f),$$

and therefore $T_1 \leqslant T_2$.

Conversely, assume that $T_1 \leqslant T_2$, and let $E \in \mathfrak{B}(X)$. Then there exists a monotone increasing sequence of real-valued, positive functions $\{f_i\}_{i=1}^{\infty} \subset C_0(X)$ such that $f_i \to 1_E$. It follows from the monotone convergence theorem that

$$\lim_{i\to\infty} T_j(f_i) = \lim_{i\to\infty} \int_X f_i d\mu_j = \int_X 1_E d\mu_j = \mu_j(E)$$

for $j = 1, 2$. Since $T_1 \leqslant T_2$, $T_1(f_i) \leqslant T_2(f_i)$ for all i. It follows that $\mu_1(E) \leqslant \mu_2(E)$. This proves our theorem. □

1.5. Product Measures and Fubini's Theorem

Let X_1 and X_2 be locally compact Hausdorff spaces and ι_1 and ι_2 positive, σ-finite Borel measures on X_1 and X_2, respectively. Then $X_1 \times X_2$ is also a locally compact Hausdorff space under the product topology, and we define the product measure $\iota_1 \times \iota_2$ in the usual way:

$$\iota_1 \times \iota_2(U) = \tag{14}$$
$$\inf \left\{ \sum_{i=1}^{n} \mu(A_i)\mu(B_i) : A_i \in \mathfrak{B}(X_1),\ B_i \in \mathfrak{B}(X_2),\ U \subset \bigcup_{i=1}^{n} A_i \times B_i \text{ disjoint union} \right\}.$$

The measure $\iota_1 \times \iota_2$ is a Borel measure on $X_1 \times X_2$, and $\iota_1 \times \iota_2(A \times B) = \iota_1(A)\iota_2(B)$ for any $A \in \mathfrak{B}(X_1)$ and $B \in \mathfrak{B}(X_2)$. Since ι_1 and ι_2 are σ-finite, so too is $\iota_1 \times \iota_2$, and it is the unique σ-finite measure satisfying $\iota_1 \times \iota_2(A \times B) = \iota_1(A)\iota_2(B)$. Moreover, $\iota_1 \times \iota_2(X_1 \times X_2) = \iota_1(X_1)\iota_2(X_2)$, so $\iota_1 \times \iota_2$ is finite if and only if ι_1 and ι_2 are both finite. Before continuing, we recall an important result from measure theory.

Theorem 29 (Fubini)**.** Suppose that ι_1 and ι_2 are positive, σ-finite measures on X_1 and X_2, respectively, and let $f \in L^1(\iota_1 \times \iota_2)$. Define $f(\cdot, y) : X_1 \mapsto \mathbb{C}$ and $f(x, \cdot) : X_2 \mapsto \mathbb{C}$ by $f(\cdot, y) : x \mapsto f(x, y)$ and $f(x, \cdot) : y \mapsto f(x, y)$. Then $f(\cdot, y) \in L_1(\iota_1)$ for ν-a.e. y and $f(x, \cdot) \in L_1(\iota_2)$ for μ-a.e. x. The functions $y \mapsto \int f(\cdot, y) d\mu$ and $x \mapsto \int f(x, \cdot) d\nu$ are defined ι_2 and ι_1-a.e., respectively. Define $F_1(x)$ to be $\int f(x, y) d\iota_2(y)$ whenever the integral exists and 0 otherwise. Define $F_2(y)$ to be $\int f(x, y) d\iota_1(x)$ whenever the integral exists and 0 otherwise. Then $F_1 \in L_1(\iota_1)$ and $F_2 \in L_2(\iota_2)$ and

$$\int_{X \times Y} f d\iota_1 \times \iota_2 = \int_X F_1(x) d\iota_1(x) = \int_{X_2} F_2(y) d\iota_2(y).$$

Proof. See, for example, [3] pp. 64-67. □

Since a measure $\mu \in \mathbf{M}(X)$ can be expressed as a linear combination of finite, positive measures, we will be able to define the product of complex measures.

Definition 30. Let $\mu \in \mathbf{M}(X_1)$ and $\nu \in \mathbf{M}(X_2)$. Define $\mu_1 = \mu_I^+$, $\mu_2 = \mu_R^-$, $\mu_3 = \mu_I^-$, and $\mu_4 = \mu_R^+$. Also define $\nu_1 = \nu_I^+$, $\nu_2 = \nu_R^-$, $\nu_3 = \nu_I^-$, and $\nu_4 = \nu_R^+$. Then $\mu = \sum_{j=1}^4 (i)^j \mu_j$ and $\nu = \sum_{j=1}^4 (i)^j \nu_j$. We define $\mu \times \nu$ to be the Borel measure on $X_1 \times X_2$ given by

$$\mu \times \nu = \sum_{j,k=1}^{4} (i)^{j+k} \mu_j \times \nu_k. \tag{15}$$

Since $\mu \times \nu$ is a linear combination of finite measures, it too must be finite, so $\mu \times \nu \in \mathbf{M}(X \times Y)$. From the definition, for any $A \in \mathfrak{B}(X_1)$ and $B \in \mathfrak{B}(X_2)$ we have the identity

$$\mu(A\times B)=\sum_{j,k=1}^{4}(i)^{j+k}\mu_j\times\nu_k(A\times B)=\sum_{j,k=1}^{4}(i)^{j+k}\mu_j(A)\nu_k(B)=\mu(A)\nu(B).$$

The product of complex measures also preserves the total variation in a certain sense.

Theorem 31. Let $\mu \in \mathbf{M}(X_1)$ and $\nu \in \mathbf{M}(X_2)$. Then $|\mu \times \nu| = |\mu| \times |\nu|$.

Proof. Let $A \in \mathfrak{B}(X_1)$ and $B \in \mathfrak{B}(X_2)$, and consider $1_{A\times B}(x,y) = 1_A(x)1_B(y)$. By Fubini's theorem,

$$\begin{aligned}\int_{X_1\times X_2} & 1_{A\times B}(x,y)\frac{d\mu}{d|\mu|}(x)\frac{d\nu}{d|\nu|}(y)d|\mu|\times|\nu|(x,y)\\ &=\int_{X_1}1_A(x)\frac{d\mu}{d|\mu|}(x)d|\mu|(x)\int_{X_2}1_B(y)\frac{d\nu}{d|\nu|}(y)d|\nu|(y)\\ &=\mu(A)\nu(B)=\int_{X_1\times X_2}1_{A\times B}d\mu\times\nu.\end{aligned}$$

Define $g(x,y) = \frac{d\mu}{d|\mu|}(x)\frac{d\nu}{d|\nu|}(y)$. Then $|g(x,y)| = 1$ for $|\mu|$-a.e. x and $|\nu|$-a.e. y. It follows $|g(x,y)| = 1$ for $|\mu| \times |\nu|$-a.e (x,y). Since the algebra of all sets of the form $A \times B$ with $A \in \mathfrak{B}(X_1)$ and $B \in \mathfrak{B}(X_2)$ generates $\mathfrak{B}(X_1 \times X_2)$, we may conclude $d\mu \times \nu = gd|\mu| \times |\nu|$. It follows from Lemma (9) that $d|\mu \times \nu| = |g|d|\mu| \times |\nu| = d|\mu| \times |\nu|$. □

Using this result, we can prove a complex verision of Fubini's theorem.

Theorem 32 (Fubini)**.** Let $\mu \in \mathbf{M}(X_1)$ and $\nu \in \mathbf{M}(X_2)$, and $f \in L_1(|\mu \times \nu|)$. Define $f(\cdot,y) : X_1 \mapsto \mathbb{C}$ and $f(x,\cdot) : X_2 \mapsto \mathbb{C}$ by $f(\cdot,y) : x \mapsto f(x,y)$ and $f(x,\cdot) : y \mapsto f(x,y)$. Then $f(\cdot,y) \in L_1(|\mu|)$ for ν-a.e. y and $f(x,\cdot) \in L_1(|\nu|)$ for μ-a.e. x. The functions $y \mapsto \int f(\cdot,y)d\mu$ and $x \mapsto \int f(x,\cdot)d\nu$ are defined ν and μ-a.e., respectively. Define $F_1(x)$ to be $\int f(x,y)d\nu(y)$ whenever the integral exists and 0 otherwise. Define $F_2(y)$ to be $\int f(x,y)d\mu(x)$ whenever the integral exists and 0 otherwise. Then $F_1 \in L_1(|\mu|)$ and $F_2 \in L_2(|\nu|)$ and

$$\int_{X_1 \times X_2} f d\mu \times \nu = \int_X F_1(x) d\mu(x) = \int_{X_2} F_2(y) d\nu(y).$$

Proof. All the statements in the theorem are simply results of Fubini's theorem for positive measures applied to $|\mu|$ and $|\nu|$, except for the last one. From the proof of Theorem (31), we know that $d\mu \times \nu = gd|\mu| \times |\nu|$ with $g(x,y) = \frac{d\mu}{d|\mu|}(x)\frac{d\nu}{d|\nu|}(y)$. Thus by Fubini's theorem

$$\begin{aligned}
\int_{X_1 \times X_2} f d\mu \times \nu &= \int_{X_1 \times X_2} f(x,y)\frac{d\mu}{d|\mu|}(x)\frac{d\nu}{d|\nu|}(y) d|\mu| \times |\nu|(x,y) \\
&= \int_{X_1}\int_{X_2} f(x,y)\frac{d\mu}{d|\mu|}(x)\frac{d\nu}{d|\nu|}(y) d|\mu|(x) d|\nu|(y) \\
&= \int_{X_1}\int_{X_2} f(x,y) d\mu(x) d\nu(y).
\end{aligned}$$

Similarly, $\int_{X_1 \times X_2} f d\mu \times \nu = \int_{X_1}\int_{X_2} f(x,y) d\nu(y) d\mu(x)$. □

1.6. The $*$-Algebra $\mathbf{M}(G)$

Let G be a T_0 topological group, and let $\mathcal{F}(G)$ be the collection of all complex-valued functions on G. Then G has continuous left and right translation operations, which map Borel sets to Borel sets. These operations give rise to a natural convolution operations on the linear space $C_0^*(G)$. Moreover, using inversion we may define a natural adjoint operation on $C_0^*(G)$. Thus we may consider $C_0^*(G)$, and by extension $\mathbf{M}(G)$, as a normed $*$-algebra. In this section, we develop the theory necessary to prove $\mathbf{M}(G)$ is a $*$-algebra. In the next section, we pick up the topic of the adjoint operation on $\mathbf{M}(G)$. We begin with some basic definitions.

Definition 33. An *algebra* A over a field k is a vector space over k with an additional binary operation of vector multiplication $A \times A \to A : (x, y) \mapsto x * y$, which satisfies the properties

(i) $(x + y) * z = x * z + y * z$

(ii) $z * (x + y) = z * x + z * y$

(iii) $(\alpha x) * (\beta y) = \alpha\beta(x * y)$

for all $x, y, z \in A$ and $\alpha, \beta \in k$.

Unless otherwise stated, by an algebra we will always mean an algebra over the field of complex numbers $\mathbb{C}$.

Definition 34. Let A be an algebra. A surjective mapping $x \mapsto x^*$ on A satisfying the properties

(i) $(x + y)^* = x^* + y^*$

(ii) $(\alpha x)^* = \overline{\alpha} x^*$

(iii) $(xy)^* = y^* x^*$

(iv) $x^{**} = x$

for all $x, y \in A$ and $\alpha \in \mathbb{C}$ is called an *adjoint operation* on A. An algebra with an adjoint operation is called a $*$-*algebra*. An element x of a $*$-algebra satisfying $x^* = x$ is called *Hermitian*. If A is a normed algebra with an adjoint operation and $\|x^*\| = \|x\|$ for all $x \in A$, then A is called a *normed* $*$-*algebra*. A Banach algebra that is a normed $*$-algebra is a *Banach* $*$-*algebra*.

To define the convolution operation, we first require some preliminary results.

Definition 35. For any function $f \in \mathcal{F}(G)$ and $a \in G$, define $L(a)f$ to be the function defined by $L(a)f(x) = f(a^{-1}x)$[4]. For any $a \in G$, the function $L(a)$ is a linear operator on the complex vector space $\mathcal{F}(G)$. The function $L : G \to \mathrm{Hom}(\mathcal{F}(G), \mathcal{F}(G))$ is called the *left regular representation* of $\mathcal{F}(G)$.

Since $L(ab)f(x) = f(b^{-1}a^{-1}x) = L(a)f(b^{-1}x) = L(a)L(b)f(x)$ for all $a, b \in G$, the left regular representation is a homomorphism of G into $\mathrm{Hom}(\mathcal{F}(G), \mathcal{F}(G))$. For any $a \in G$ the translation operation $x \mapsto a^{-1}x$ is continuous. Thus the restriction of $L(a)$ to $C(G)$, $C_0(G)$, or $C_{00}(G)$ is a linear operator on that linear subspace.

Lemma 36. Let $T \in C_0^*(G)$. Then for any $f \in C_0(G)$, the function $x \mapsto \langle L(x^{-1})f, T\rangle$ is in $C_0(G)$.

Proof. Let $T \in C_0^*(G)$ and $f \in C_0$. We must verify two things: the function defined above is continuous, the function can be made arbitrarily small outside a compact set. We begin by proving continuity. For all $\epsilon > 0$, there exists an open neighborhood U of the identity $e \in G$ such that for all $x, y \in G$, $xy^{-1} \in U$ implies $|f(x) - f(y)| < \epsilon/\|T\|$. It follows that $\|L(x^{-1})f - L(y^{-1})f\|_u < \epsilon/\|T\|$, and therefore $|\langle L(x^{-1})f, T\rangle - \langle L(y^{-1})f, T\rangle| = |\langle L(x^{-1})f - L(y^{-1})f, T\rangle| < \epsilon$. This proves continuity.

Let $\epsilon > 0$, and let $\mu \in \mathbf{M}(G)$ be the measure satisfying $\Phi(\mu) = T$. To complete the proof, we must show that $\{x : |f(x)| \geqslant \epsilon\}$ is contained in a compact set. There exist compact subsets $C_1, C_2 \subset G$ such that $|f(x)| < \epsilon/2\|\mu\|$ for all $x \notin C_1$ and $|\mu|(C_2^c) < \epsilon/2\|f\|_u$. It follows that

$$|\langle L(a^{-1})f, T\rangle| = \left|\int_G f(ax)d\mu(x)\right| \leqslant \int_G |f(ax)|d|\mu|(x) < \int_{C_2} |f(ax)|d|\mu|(x) + \epsilon/2$$

If $a \notin C_1C_2^{-1}$ and $x \in C_2$, then $ax \notin C_1$ and therefore $|f(ax)| < \epsilon/2\|\mu\|$. Thus

[4]This operation may look a little strange at first: why not instead define $L(a)f(x) = f(ax)$, as this would seem to be the simpler and more intuitive thing to do. The answer is that L would not be a homomorphism under this definition.

the above inequality tells us $|\langle L(a^{-1})f, T\rangle| < \epsilon$ for $a \notin C_1C_2^{-1}$. Since the mapping $x \mapsto x^{-1}$ is continuous, C_2^{-1} is compact. The product of compact sets is compact, so $C_1C_2^{-1}$ is compact. This proves that the function defined above is less than ϵ outside a compact set. Since $\epsilon > 0$ was arbitrary, this proves our theorem. □

Using this lemma, we can now define the convolution of two linear functionals in $C_0^*(G)$.

Definition 37. Let $T_1, T_2 \in C_0^*(G)$. The *convolution of functionals* $T_1 * T_2$ is the linear functional on $C_0(G)$ defined by

$$\langle f, T_1 * T_2\rangle = \langle\langle L(\cdot^{-1})f, T_2\rangle, T_1\rangle,$$

where $\langle L(\cdot^{-1})f, T_2\rangle$ is the function in $C_0(G)$ defined by $x \mapsto \langle L(x^{-1})f, T_2\rangle$.

The next theorem shows that the convolution of functionals is a linear functional and that $C_0^*(G)$ is an algebra

Theorem 38. Let $T_1, T_2 \in C_0^*(G)$. Then $T_1 * T_2 \in C_0^*(G)$ and $\|T_1 * T_2\| \leqslant \|T_1\|\|T_2\|$. The complex linear space $C_0^*(G)$ is an algebra with respect to this convolution operation.

Proof. For any $f \in C_0(G)$ and $T \in C_0^*(G)$, let $\langle L(\cdot^{-1})f, T\rangle$ be the function in $C_0(G)$ defined by $x \mapsto \langle L(x^{-1})f, T_2\rangle$ and let $L(\cdot^{-1})f$ be the function defined by $x \mapsto L(x^{-1})f$. Let $f_1, f_2, f \in C_0(G)$ and $\alpha \in C$. Then $L(\cdot^{-1})(\alpha f) = \alpha L(\cdot^{-1})f$ and $L(\cdot^{-1})(f_1 + f_2) = L(\cdot^{-1})f_1 + L(\cdot^{-1})f_2$. It follows that $\langle \alpha L(\cdot^{-1})f, T\rangle = \alpha\langle L(\cdot^{-1})f, T\rangle$, and $\langle L(\cdot^{-1})(f_1 + f_2), T\rangle = \langle L(\cdot^{-1})f_1, T\rangle + \langle L(\cdot^{-1})f_2, T\rangle$. The linearity of $T_1 * T_2$ follows immediately. Moreover

$$\langle\langle L(\cdot^{-1})f, T_2\rangle, T_1\rangle \leqslant \|T_1\|\|\langle L(\cdot^{-1})f, T_2\rangle\|_u \leqslant \|T_1\|\|T_2\|\|f\|_u,$$

and therefore $T_1 * T_2 \in C_0^*(G)$ with $\|T_1 * T_2\| \leqslant \|T_1\|\|T_2\|$.

Additionally for any $T \in C_0^*(G)$ and any $\alpha \in \mathbb{C}$, $\langle L(\cdot^{-1})f, (T+T_2)\rangle = \langle L(\cdot^{-1})f, T\rangle + \langle L(\cdot^{-1})f, T_2\rangle$, and $\langle L(\cdot^{-1})f, \alpha T_2\rangle = \alpha\langle L(\cdot^{-1})f, T_2\rangle$. Thus

$$\begin{aligned}\langle f, T * (T_1 + T_2)\rangle &= \langle\langle L(\cdot^{-1})f, (T_1 + T_2)\rangle, T\rangle = \langle\langle L(\cdot^{-1})f, T_1\rangle + \langle L(\cdot^{-1})f, T_2\rangle, T\rangle \\ &= \langle\langle L(\cdot^{-1})f, T_1\rangle, T\rangle + \langle\langle L(\cdot^{-1})f, T_2\rangle, T\rangle = \langle f, T * T_1\rangle + \langle f, T * T_2\rangle\end{aligned}$$

and

$$T_1 * (\alpha T_2) = \langle\langle L(\cdot^{-1})f, \alpha T_2\rangle, T_1\rangle = \langle\alpha\langle L(\cdot^{-1})f, T_2\rangle, T_1\rangle = \alpha\langle\langle L(\cdot^{-1})f, T_2\rangle, T_1\rangle$$

Similarly $(T_1 + T_2) * T = T_1 * T + T_2 * T$ and $(\alpha T_1) * T_2 = \alpha T_1 * T_2$. It follows that $C_0^*(G)$ is an algebra. This proves our theorem. □

The isometry Φ and the convolution on $C_0^*(G)$ gives us a readily available means of defining the convolution of measures.

Definition 39. Let $\mu, \nu \in \mathbf{M}(G)$, and let $\Phi(\mu) = T_\mu$ and $\Phi(\nu) = T_\nu$. We define the *convolution of measures* $\mu * \nu$ to be the measure in $\mathbf{M}(G)$ such that $\Phi(\mu * \nu) = T_\mu * T_\nu$.

Theorem 40. Let $\mu_1, \mu_2 \in \mathbf{M}(G)$. Then $|\mu_1 * \mu_2| \leqslant |\mu_1| * |\mu_2|$.

Proof. Let $T_i \in C_0^*(G)$ be the linear functional corresponding to μ_i for $i = 1, 2$. Let $f \in C_0(G)$ be a real-valued, positive function on G and let $\varphi \in C_0(G)$ with $|\varphi| \leqslant f$. Then we have that

$$\begin{aligned}|\langle\varphi, T_1 * T_2\rangle| = \left|\int_G\int_G \varphi(xy)d\mu_2(y)d\mu_1(x)\right| &\leqslant \int_G\left|\int_G \varphi(xy)d\nu(y)\right|d|\mu|(x) \\ &\leqslant \int_G\int_G |\varphi(xy)|d|\nu|(y)d|\mu|(x) \\ &\leqslant \int_G\int_G f(xy)d|\nu|(y)d|\mu|(x) = \langle f, |T_1| * |T_2|\rangle.\end{aligned}$$

Taking the supremum over all $\varphi \in C_0(G)$ with $|\varphi| \leqslant f$, we find $|T_1 * T_2| \leqslant |T_1| * |T_2|$. The isomorphism Φ is order-preserving, and therefore $|\mu_1 * \mu_2| \leqslant |\mu_1| * |\mu_2|$. □

The next theorem provides an alternative definition of the convolution of measures which turns out to be equivalent. Moreover, it describes for us a means of evaluating $\int_G f d\mu_1 * \mu_2$ in the case that $f \in L_1(|\mu_1| * |\mu_2|)$.

Theorem 41. Let $\mu_1, \mu_2 \in \mathbf{M}(G)$ and let η be the Borel measure on G defined by $\eta(E) = \mu_1 \times \mu_2(\tau^{-1}(E))$ for all $E \in \mathfrak{B}(G)$. Then $\eta = \mu_1 * \mu_2$ and for any Borel-measureable function f with $f \circ \tau \in L_1(|\mu_1| \times |\mu_2|)$,

$$\begin{aligned}\int_G f d\mu_1 * \mu_2 &= \int_G f d\eta = \int_G f \circ \tau d\mu_1 \times \mu_2 \\ &= \int_G \int_G f(xy) d\mu_2(y) d\mu_1(x) = \int_G \int_G f(xy) d\mu_1(x) d\mu_2(y).\end{aligned} \tag{16}$$

Moreover, $f \in L_1(|\mu_1| * |\mu_2|)$ implies $f \circ \tau \in L_1(|\mu_1| \times |\mu_2|)$.

Proof. Let $T_i \in C_0^*(G)$ be the linear functional corresponding to μ_i for $i = 1, 2$. If $f \in C_0(G)$, then $f \circ \tau \in C_0(G)$, and therefore $f \circ \tau \in L_1(|\mu_1| \times |\mu_2|)$. Thus by Fubini's theorem,

$$T_1 * T_2(f) = \int_G \int_G f(xy) d\mu_2(y) d\mu_1(x) = \int_{G \times G} f \circ \tau d\mu_1 \times \mu_2.$$

The mapping $f \mapsto \int_{G \times G} f \circ \tau d\mu_1 \times \mu_2$ is a linear functional on $C_0(G)$. Moreover, by a similar argument, Fubini's theorem tells us

$$\begin{aligned}\left| \int_{G \times G} f \circ \tau d\mu_1 \times \mu_2 \right| &\leqslant \int_{G \times G} |f \circ \tau| d|\mu_1 \times \mu_2| \\ &= \int_{G \times G} |f| \circ \tau d|\mu_1| \times |\mu_2| = \int_{G \times G} |f| \circ \tau d|\mu_1| \times |\mu_2| \\ &= \int_G |f| d|\mu_1| * |\mu_2| = |T_1| * |T_2|(|f|) \leqslant \|T_1\| \|T_2\| \|f\|_u.\end{aligned}$$

It follows that $f \mapsto \int_{G\times G} f\circ\tau d\mu_1\times\mu_2$ is a bounded linear functional on $C_0(G)$, and therefore is the same as $T_1 * T_2$. Now given any $E\in\mathfrak{B}(G)$, $\tau^{-1}(E)\in\mathfrak{B}(G\times G)$, since τ is continuous. Also, $\tau(x,y)\in E$ if and only if $(x,y)\in\tau^{-1}(E)$. It follows that $1_E\circ\tau = 1_{\tau^{-1}(E)}$ and therefore $\eta(E)=\int_G 1_E\circ\tau d\mu_1\times\mu_2=\int_G 1_{\tau^{-1}(E)}d\mu_1\times\mu_2 = \mu_1\times\mu_2(\tau^{-1}(E))$. From this, we may conclude that the equality

$$\int_G f\circ\tau d\mu_1\times\mu_2=\int_G f d\eta$$

holds for all simple Borel measureable functions f, and therefore for all Borel measureable functions f with $f\circ\tau\in L_1(|\mu_1|\times|\mu_2|)$. Since Φ is an isomorphism, we may conclude $\eta=\mu_1*\mu_2$. This combined with Fubini's theorem proves Equation (16).

Next, consider a compact subset $C\subset G$, and let $\epsilon>0$. Then there exists an open subset $U\subset G$ such that $C\subset U$ and $|\mu|*|\nu|(U)-|\mu|*|\nu|(C)<\epsilon$. Let $f\in C_0(G)$ be a function with support in U, with $f(x)=1$ for all $x\in C$, and with $0\leqslant f\leqslant 1$. Then by Fubini's theorem

$$\begin{aligned}\int_{G\times G} 1_C\circ\tau d|\mu_1|\times|\mu_2| &= \int_G\int_G 1_C(xy)d|\mu_2|(y)d|\mu_1|(x)\\ &\leqslant \int_G\int_G f(xy)d|\mu_2|(y)d|\mu_1|(x) = |T_1|*|T_2|(f)\\ &= \int_G f d|\mu_1|*|\mu_2| \leqslant \int_G 1_U d|\mu_1|*|\mu_2|\\ &= |\mu_1|*|\mu_2|(U) < |\mu_1|*|\mu_2|(C)+\epsilon.\end{aligned}$$

Since $\epsilon>0$ was arbitrary, it follows that the inequality

$$\int_{G\times G} 1_C f d|\mu_1|\times|\mu_2| \leqslant |\mu_1|*|\mu_2|(C),$$

for all compact sets, and therefore for all σ-compact Borel sets. Moreover, if $O\in$

$\mathfrak{B}(G)$ is a set of $|\mu_1| * |\mu_2|$-measure zero, then for any compact subset $C \subset \tau^{-1}(O)$, $\tau(C)$ is compact and $1_C \leqslant 1_{\tau(C)} \circ \tau$. It follows that

$$|\mu_1| \times |\mu_2|(C) \leqslant \int_G 1_{\tau(C)} \circ \tau d|\mu_1| \times |\mu_2| \leqslant |\mu_1| * \mu_2|(C) \leqslant |\mu_1| * |\mu_2|(O) = 0.$$

Since this is true of any compact subset of $\tau^{-1}(O)$, it follows that

$$\int_G 1_O \circ \tau d|\mu_1| \times |\mu_2| = |\mu_1| \times |\mu_2|(\tau^{-1}(O)) = 0.$$

Now given any $E \in \mathfrak{B}(G)$, since $|\mu| * |\nu|$ is σ-finite, E may be written as the union of disjoint sets $E = C \cup O$, where C is σ-compact and O is a set of $|\mu| * |\nu|$-measure 0. It follows that

$$\begin{aligned}\int_G 1_E \circ \tau d|\mu_1| \times |\mu_2| &= \int_G 1_C \circ \tau d|\mu_1| \times |\mu_2| + \int_G 1_O \circ \tau d|\mu_1| \times |\mu_2| \\ &\leqslant |\mu| * |\nu|(C) + |\mu| * |\nu|(O) = |\mu| * |\nu|(E).\end{aligned}$$

The inequality

$$\int_G f \circ \tau d|\mu_1| \times |\mu_2| \leqslant \int_G f d|\mu| * |\nu|(E)$$

then follows immediately for simple functions and subsequently for positive Borel-measureable functions by the monotone convergence theorem. This in turn guarantees it for all $f \in L_1(|\mu| * |\nu|)$, proving our theorem. □

The above theorem provides us with the most direct means of actually evaluating the convolution of two measures, as the next examples demonstrate.

Example 42. Let $G = R$ and let $a > 0$. Let μ be the measure defined by $d\mu = \frac{1}{2a} 1_{[-a,a]} dx$, where dx represents the usual Lebesgue measure on R. Then by Theorem

(41) below and Fubini's theorem,

$$\begin{aligned}
\mu * \mu((-\infty, b]) &= \int_{R\times R} 1_{(-\infty,b]}(x+y)d\mu(x) \times d\mu(y) \\
&= \frac{1}{4a^2}\int_{-a}^{a}\int_{-a}^{a} 1_{(-\infty,b]}(x+y)dydx \\
&= \frac{1}{4a^2}\int_{-a}^{a}\int_{x-a}^{x+a} 1_{(-\infty,b]}(y)dydx \\
&= \begin{cases} 0, & b \leqslant -2a \\ \frac{1}{8a^2}(2a+b)^2, & -2a \leqslant b \leqslant 0; \\ 1 - \frac{1}{8a^2}(2a-b)^2, & 0 \leqslant b \leqslant 2a; \\ 1, & b \geqslant 2a. \end{cases}
\end{aligned}$$

It follows that $d\mu * \mu = (1 - |x|/(2a))1_{[-2x,2x]}(x)dx$.

Example 43. Let G be the group of all 2×2 matrices of the form

$$(x, y) := \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix},$$

with $x \neq 0$. Then $(a, b)(x, y) = (ax, ay + bx^{-1})$ and G is a topological group under the metric defined by $\rho((x, y), (x', y')) = \sqrt{(x - x')^2 + (y - y')^2}$ and has a left Haar measure λ defined by $d\lambda(x, y) = \frac{1}{x^2}dm(x, y)$, where m is the usual Lebesgue measure on $(\mathbb{R}\backslash\{0\}) \times \mathbb{R}$. If $f \in L_1(\lambda)$, then the measure μ defined by $d\mu = fd\lambda$ is a complex measure absolutely continuous with respect to λ. Let $\delta_{(a,b)}$ be the dirac measure at (a, b). Then $d\delta_{(a,b)} * \mu = L((a, b))fd\lambda = f((a^{-1}, -b)(x, y))d\lambda(x, y)$.

1.7. Decomposition of $\mathbf{M}(G)$ and Adjoint Operators

In this section, we decompose the space $\mathbf{M}(G)$ into a direct sum of linear subspaces and in particular prove that the collection $\mathbf{M}_a(G)$ of measures absolutely continuous with respect to a left Haar measure λ on G is an ideal in $\mathbf{M}(G)$. Thus, given any

$\mu \in \mathbf{M}(G)$ and $\nu \in \mathbf{M}_a(G)$, the measure $\mu * \nu \in \mathbf{M}_a(G)$, and therefore there exists a function g such that $d\mu * \nu = g d\lambda$. In particular, since $\mathbf{M}_a(G)$ is isometric to $L_1(\lambda)$, the mapping $f \mapsto \mu * f$, where $\mu * f \in L_1(G)$ is the function defined by $d\mu * \nu = \mu * f d\lambda$ with $d\nu = f d\lambda$, is a bounded linear operator on $L_1(G)$. Moreover, we show that $\mathbf{M}(G) * L_p(\lambda) \subset L_p(\lambda)$ for all $1 \leqslant p \leqslant \infty$, and therefore the mapping $f \mapsto \mu * f$ is a bounded linear operator on the Hilbert space $L_2(\lambda)$. In particular then, this demands the existence of an adjoint, which we shall define.

As a vector space, the algebra $\mathbf{M}(G)$ can be decomposed into a direct sum of linear subspaces. To begin, we have the following definitions.

Definition 44. A measure $\mu \in \mathbf{M}(G)$ is said to be *continuous* if $\mu(\{x\}) = 0$ for all $x \in G$, and the collection of all such measures is denoted by $\mathbf{M}_c(G)$. It is said to be *discrete* if the support of the measure is contained in a countable set, and the collection of all discrete measures is denoted $\mathbf{M}_d(G)$. It is *absolutely continuous* if it is continuous with respect to a left Haar measure on G, and the collection of all such measures is denoted $\mathbf{M}_a(G)$. Lastly, a measure is *singular* if it is continuous and singular with respect to a left Haar measure on G, and the collection of all such measures is denoted $\mathbf{M}_s(G)$.

It follows immediately from the definitions that $\mathbf{M}_a(G)$, $\mathbf{M}_d(G)$, and $\mathbf{M}_s(G)$ are all linear subspaces of $\mathbf{M}(G)$ which intersect only at $\{0\}$. Moreover, we have the following theorem.

Theorem 45. If G is not discrete, then $\mathbf{M}(G)$ can be decomposed into the linear sum

$$\mathbf{M}(G) = \mathbf{M}_d(G) \oplus \mathbf{M}_s(G) \oplus \mathbf{M}_a(G).$$

Thus, each $\mu \in \mathbf{M}(G)$ may be decomposed into a direct sum $\mu = \mu_d + \mu_s + \mu_a$ of discrete, singular and absolutely continuous measures and

$$|\mu| = |\mu_d| + |\mu_s| + |\mu_a|$$

and

$$\|\mu\| = \|\mu_d\| + \|\mu_s\| + \|\mu_a\|.$$

Proof. Given $\mu \in \mathbf{M}(G)$, the collection A_μ of all $x \in G$ such that $\mu(\{x\}) \neq 0$ is countable[5], since μ is finite. Define $\mu_d \in \mathbf{M}(G)$ by $\mu_d(E) = \mu(E \cap A_\mu)$ for all $E \in \mathfrak{B}(G)$. Then $\mu_d \in \mathbf{M}_d(G)$ and $\mu_c = \mu - \mu_d$ is continuous. Moreover the Lebesgue-Radon-Nikodym Theorem tells us that $\mu_c = \mu_s + \mu_a$, where μ_s is singular and μ_a is absolutely continuous. It follows that $\mu = \mu_d + \mu_s + \mu_a \in \mathbf{M}_d(G) + \mathbf{M}_s(G) + \mathbf{M}_a(G)$. Since $\mathbf{M}_a(G)$, $\mathbf{M}_d(G)$, and $\mathbf{M}_s(G)$ pairwise intersect to $\{0\}$, this proves $\mathbf{M}(G) = \mathbf{M}_d(G) \oplus \mathbf{M}_s(G) \oplus \mathbf{M}_a(G)$.. The measures μ_d, μ_a, and μ_s are all mutually singular by definition, and therefore $|\mu| = |\mu_d| + |\mu_s| + |\mu_a|$. The rest of the theorem follows immediately. □

In order to prove that the convolution of a measure in $\mu \in \mathbf{M}(G)$ with a measure ν of the form $d\nu = f d\lambda$ with $f \in L_p(G)$ is again a measure of the form $d\mu * \nu = g d\lambda$ for some $g \in L_p(G)$, we require a technical lemma.

Lemma 46. Let f be a λ-measureable function on G. Then $(x, y) \mapsto f(x^{-1}y)$ and $(x, y) \mapsto f(x)$ are $\lambda \times |\mu|$-measureable for all $\mu \in \mathbf{M}(G)$

Proof. See [2] pp. 287-88. □

Using this lemma, we have the following theorem.

[5] By a countable set, we mean a set which is either finite or countably infinite.

Theorem 47. Suppose that $1 \leqslant p, q < \infty$ with $1/p + 1/q = 1$ and that $f \in L_p(\lambda)$. Let $\kappa(x, y) = x^{-1}y$ and $\mu \in \mathbf{M}(G)$. Then if $p < \infty$, the function h defined by

$$\mu * f(x) = \int_G f(y^{-1}x)d\mu(y) \tag{17}$$

exists and is finite for λ-a.e. $x \in G$. Moreover $\mu * f \in L_p(\lambda)$, and $\|h\|_p \leqslant \|f\|_p\|\mu\|$. If $p = 1$ and $\nu \in \mathbf{M}(G)$ is the measure defined by $d\nu = fd\lambda$, then $d\mu * \nu = \mu * fd\lambda$.

Proof. By our previous lemma, we know that $f \circ \kappa$ is $\lambda \times |\mu|$-measureable. Hölder's inequality tells us that for all $h \in L_q(\lambda)$,

$$\int_G |f(x^{-1}y)h(y)|dy \leqslant \|L(x^{-1})f\|_p\|h\|_q = \|f\|_p\|h\|_q,$$

from which it follows that

$$\int_G \int_G |f(x^{-1}y)h(y)|dyd|\mu|(x) \leqslant \int_G \|f\|_p\|h\|_q d|\mu|(x) = \|f\|_p\|h\|_q\|\mu\|.$$

Thus

$$\int_G \int_G f(x^{-1}y)h(y)\frac{d\mu}{d|\mu|}(x)d|\mu|(x)dy = \int_G h(y)\left(\int_G f(x^{-1}y)\frac{d\mu}{d|\mu|}(x)d|\mu|(x)\right)dy$$

is a well-defined function in $L_1(\lambda)$ and it follows that

$$h(y)\int_G f(x^{-1}y)\frac{d\mu}{d|\mu|}(x)d|\mu|(x)$$

exists and is finite for λ-a.e. $y \in G$. We define

$$\mu * f(y) = \int_G f(x^{-1}y)\frac{d\mu}{d|\mu|}(x)d|\mu|(x) = \int_G f(x^{-1}y)d\mu(x)$$

whenever the integral exists and 0 otherwise. Since the value of h was arbitrary, necessarily $\mu * f$ is λ-measureable.

Since $f \in L_p(\lambda)$, the support of f is contained in a σ-compact subset $A \subset G$. Moreover, since $\mu \in \mathbf{M}(G)$, there exists a σ-compact subset $B \subset G$ such that $\mu(B^c) = 0$. Let $C = BA$. Then since the product of compact sets is compact, C is σ-compact. Thus there exists a sequence of functions $\{\psi_i\}_{i=1}^{\infty} \subset C_{00}(G)$ with $0 \leqslant \psi_i \leqslant 1$ for all i and $\psi_i(x) \to 1_C(x)$ for λ-a.e. $x \in G$.

$$\mu * f(y) = \lim_{n\to\infty} \psi_i(y) \int_G f(x^{-1}y) \frac{\partial \mu}{\partial |\mu|}(x) d|\mu|(x)$$

exists and is finite and equal to $\mu * f(y)$ for λ-a.e. $y \in C$. Moreover, from the definition of $\mu * f$, if $y \notin C$ then $x^{-1}y \notin A$ for all $x \in B$ and therefore $\mu * f(y) = 0$. Thus the support of $\mu * f$ is contained in a compact set and

$$\begin{aligned} \|\mu * f\|_p &= \sup\left\{ \left| \int_G \mu * f(y) h(y) dy \right| : h \in L_q(G), \|h\|_q \leqslant 1 \right\} \\ &< \|f\|_p \|h\|_q \|\mu\| \leqslant \|f\|_p \|\mu\|. \end{aligned}$$

Thus $\mu * f \in L_q(\lambda)$ and $\|\mu * f\|_p \leqslant \|f\|_p \|\mu\|$.

Lastly, let $\nu \in \mathbf{M}(G)$ be the measure defined by $d\nu = f d\lambda$. Then $\mu * f \in L_1(\lambda)$. Moreover, for any $\varphi \in C_{00}(G)$ Fubini's theorem tells us that

$$\begin{aligned} \int_G \varphi \mu * f d\lambda &= \int_G \int_G \varphi(y) f(x^{-1}y) d\mu(x) dy = \int_G \int_G \varphi(y) f(x^{-1}y) dy d\mu(x) \\ &= \int_G \int_G \varphi(xy) f(y) dy d\mu(x) = \int_G \int_G \varphi(xy) d\nu(y) d\mu(x) = \int_G \varphi d\mu * \nu.. \end{aligned}$$

Since this is true for every $\varphi \in C_{00}(G)$, the corresponding measures correspond to the same linear functional on $C_0(G)$. Thus we have that $\mu * f d\lambda = d\mu * \nu$. This proves our theorem. □

The previous theorem tells us immediately that $\mathbf{M}_a(G)$ is a left ideal of $\mathbf{M}(G)$ and allows us to accept the following definition.

Definition 48. Let $\mu \in \mathbf{M}(G)$ and $f_1, f_2 \in L_p(\lambda)$ for some $1 \leqslant p < \infty$. Moreover, let $\nu_i \in \mathbf{M}(G)$ be the measure defined by $d\nu_i = f_i d\lambda$ for $i = 1, 2$. If $p = 1$, we define $\mu * f_1, f_1 * f_2 \in L_1(\lambda)$ to be the functions satisfying $\mu * \nu_1 = \mu * f_1 d\lambda$ and $\nu_1 * \nu_2 = f_1 * f_2 d\lambda$. If $p \neq 1$, then we define $\mu * f$ as in Equation (17) of Theorem (47).

Let $\mu \in \mathbf{M}(G)$. The above theorem shows us that the function $T : f \mapsto \mu * f$ is a linear operator on $L_2(G)$. Since this operator acts on a Hilbert space, we expect there to be an adjoint operator T^* such that $\langle Tf, h\rangle = \langle f, T^*h\rangle$ for all $f, h \in L_2(\lambda)$. In particular, this means that

$$\begin{aligned}\int_G f(x)\overline{(T^*g)(x)}dx &= \int_G \mu * f(x)\overline{g(x)}dx = \int_G\int_G f(y^{-1}x)d\mu(y)\overline{g(x)}dx \\ &= \int_G\int_G f(y^{-1}x)\overline{g(x)}dxd\mu(y) = \int_G\int_G f(x)\overline{g(yx)}dxd\mu(y) \\ &= \int_G f(x)\int_G \overline{g(yx)}d\mu(y)dx.\end{aligned}$$

Since the value of $f \in L_2(\lambda)$ was arbitrary, we may conclude that

$$\overline{T^*g(x)} = \int_G \overline{g(yx)}d\mu(y).$$

Recall that $\theta : G \to G$ is the map defined by $\theta(x) = x^{-1}$. Suppose that there is a measure $\mu^* \in \mathbf{M}(G)$ such that

$$\int_G f d\mu^* = \overline{\int_G \overline{f \circ \theta} d\mu}.$$

for all $f \circ \theta \in L_1(|\mu|)$. Then

$$\mu^* * f(x) = \int_G f(y^{-1}x) d\mu^*(y) = \overline{\int_G \overline{f(\theta(y^{-1})x)} d\mu^*(y)} = T^* f,$$

and we are able to successfully express the adjoint operator T^* as a convolution with a measure. As it turns out, for all $\mu \in \mathbf{M}(G)$ there is a measure $\mu^* \in \mathbf{M}(G)$, satisfying this property, and this defines an adjoint on $\mathbf{M}(G)$ (and simultaneously on $C_0^*(G)$).

Definition 49. Let $\mu \in \mathbf{M}(G)$ and let $T \in C_0^*(G)$ with $\Phi(\mu) = T$. We define the *adjoint of* T to be the linear functional $T^* \in C_0^*(G)$ given by $T^*(\varphi) = \overline{T(\overline{\varphi \circ \theta})}$ for all $\varphi \in C_0(G)$. We define the *adjoint of* μ to be the measure μ^* satisfying $\Phi(\mu^*) = T^*$.

Since the composition $\varphi \circ \theta$ of a function $\varphi \in C_0(G)$ with the continuous function θ is in $C_0(G)$, it is obvious from the definition that T^* is a bounded linear functional on $C_0(G)$. Moreover, the adjoint operation $T \mapsto T^*$ establishes $C_0^*(G)$ as a $*$-algebra. In particular, we have the following theorem.

Theorem 50. Under the operation adjoint operation, $C_0^*(G)$ is a normed $*$-algebra. Furthermore, $|T|^* = |T^*|$ for all $T \in C_0^*(G)$.

Proof. Let $T, T_1, T_2 \in \mathbb{C}_0^*(G)$. It is clear from the definition that $(T_1 + T_2)^* = T_1^* + T_2^*$, $(\alpha T)^* = \overline{\alpha} T^*$, and $T^{**} = T$. Given any $\varphi \in C_0(G)$, $\varphi \circ \theta \in C_0(G)$ with $\|\varphi \circ \theta\|_u = \|\varphi\|_u$. Moreover, $\varphi \circ \theta \circ \theta = \varphi$ and therefore $\{\varphi : \varphi \in C_0(G),\ |\varphi| \leqslant 1\} = \{\varphi \circ \theta : \varphi \in C_0(G) : |\varphi| \leqslant 1\}$. Moreover $\{\varphi : \varphi \in C_0(G),\ |\varphi| \leqslant 1\} = \{\overline{\varphi} : \varphi \in C_0(G),\ |\varphi| \leqslant 1\}$, and therefore $\{\varphi : \varphi \in C_0(G),\ |\varphi| \leqslant 1\} = \{\overline{\varphi \circ \theta} : \varphi \in C_0(G) : |\varphi| \leqslant 1\}$. It follows that

$$\begin{aligned}
\|T^*\| &= \sup\{\|T^*(\varphi)\|_u : \varphi \in C_0(G),\ |\varphi| \leqslant 1\} \\
&= \sup\{\|T(\overline{\varphi \circ \theta})\|_u : \varphi \in C_0(G),\ |\varphi| \leqslant 1\} \\
&= \sup\{\|T(\varphi)\|_u : \varphi \in C_0(G),\ |\varphi| \leqslant 1\} = \|T\|.
\end{aligned}$$

Therefore the adjoint operation perserves the norm on $C_0^*(G)$.

Let $\mu_i \in \mathbf{M}(G)$ be the measures satisfying $\Phi(\mu_i) = T_i$ for $i = 1, 2$. Given any $\varphi \in C_0(G)$, Fubini's theorem tells us that

$$\begin{aligned}(T_1 * T_2)^*(\varphi) &= \overline{(T_1 * T_2)(\overline{\varphi \circ \theta})} = \overline{\int_G \int_G \overline{\varphi \circ \theta(xy)} d\mu_2(y) d\mu_1(x)} \\ &= \overline{\int_G \int_G \overline{\varphi(y^{-1}x^{-1})} d\mu_2(y) d\mu_1(x)} = \overline{\int_G \int_G \overline{\varphi(y^{-1}x^{-1})} d\mu_1(x) d\mu_2(y)} \\ &= \overline{\int_G \int_G \overline{L(y^{-1})(\varphi \circ \theta)(x)} d\mu_1(x) d\mu_2(y)} = \overline{\int_G \overline{T_1^*(L(y^{-1})\varphi)} d\mu_2(y)} \\ &= \overline{\int_G \overline{T_1^*(L(\theta(y))\varphi)} d\mu_2(y)} = T_2^* * T_1^*(\varphi).\end{aligned}$$

Since ϕ was arbitrary, this shows us that $(T_1 * T_2)^* = T_2^* * T_1^*$. This proves that $\mathbb{C}_0^*(G)$ is a normed $*$-algebra.

The last thing that we wish to show is $|T|^* = |T^*|$. For any real, positive function $\varphi \in C_0(G)$ we have that $|T|^*(\varphi) = |T|(\varphi \circ \theta)$. Moreover if $\psi \in C_0(G)$ with $|\psi| \leqslant \varphi$, then $|\overline{\psi \circ \theta}| \leqslant |\psi \circ \theta| \leqslant \varphi \circ \theta$. Thus

$$\begin{aligned}|T^*|(\varphi) &= \sup\{|T^*(\psi)| : \psi \in C_0(G),\ |\psi| \leqslant \varphi\} \\ &= \sup\{|T(\overline{\psi \circ \theta})| : \psi \in C_0(G),\ |\psi| \leqslant \varphi\} \\ &\leqslant \sup\{|T(\psi)| : \psi \in C_0(G),\ |\psi| \leqslant \varphi \circ \theta\} = |T|(\varphi \circ \theta) = |T|^*(\varphi).\end{aligned}$$

It follows that $|T^*| \leqslant |T|^*$. It is also clear from the definition that for real, positive linear functionals T_1, T_2, $T_1 \leqslant T_2$ implies $T_1^* < T_2^*$ and vise-versa. Consequently,

$$|T|^* = |(T^*)^*|^* \leqslant |T^*|^{**} \leqslant |T^*|.$$

Combining these inequalities, we conclude $|T|^* = |T^*|$. □

As with all the other operations on $C_0^*(G)$, there is a strictly measure theoretic interpretation. This is discussed in the next theorem. In particular, the next theorem shows that μ^* is the same measure we found during our search for the adjoint the linear transformation $f \mapsto \mu * f$ on $L_2(\lambda)$.

Theorem 51. Let $T \in C_0^*(G)$ and let μ be the measure on $\mathbf{M}(G)$ with $\Phi(\mu) = T$. Then for every $E \in \mathfrak{B}(G)$, we have $\mu^*(E) = \overline{\mu(\theta(E))}$ and $|\mu^*|(B) = |\mu|(\theta(B))$. Moreover $f \in L_1(|\mu^*|)$ if and only if $f \circ \theta \in L_1(|\mu|)$. In this case,

$$\int_G f d\mu^* = \overline{\int_G \overline{f \circ \theta} d\mu}.$$

If $f \in L_p(G)$ with $1 \leqslant p < \infty$, then

$$\mu^* * f(x) = \overline{\int_G \overline{f(yx)} d\mu(y)}$$

for λ-a.e. $x \in G$.

Proof. Since $\mu^* \in \mathbf{M}(G)$, is finite, for any open subset $U \subset G$, there exists an increasing sequence of functions $\{\psi_i\}_{i=1}^{\infty} \subset C_{00}(G)$ with $0 \leqslant \psi_i \leqslant 1$ and $\psi_i \to 1_U$ pointwise $|\mu^*|$-a.e.. Similarly, there exists an increasing sequence of functions $\{\varphi_i\}_{i=1}^{\infty} \subset C_{00}(G)$ with $0 \leqslant \varphi_i \leqslant 1$ and $\psi_i \to 1_{\theta(U)}$ pointwise $|\mu|$-a.e.. Define $\omega_i(x) = \max\{\psi_i(x), \varphi_i(x^{-1})\}$ for all $x \in G$. Then $\omega_i \to 1_U$ pointwise $|\mu^*|$-a.e. and $\omega_i^* \to 1_{\theta(U)}$ pointwise $|\mu|$-a.e.. Thus by the dominated convergence theorem,

$$\mu^*(U) = \lim_{i\to\infty} \int_G \omega_i d\mu^* = \lim_{i\to\infty} \overline{\int_G \overline{\omega_i \circ \theta} d\mu} = \lim_{i\to\infty} \overline{\int_G \omega_i d\mu} = \overline{\mu(U^{-1})}.$$

Since this is true of all open sets, the identity $\mu^*(E) = \overline{\mu(\theta(E))}$ follows immediately for all Borel sets E.

The identity $\int_G \varphi d\mu^* = \overline{\int_G \overline{f \circ \theta} d\mu}$ then follows immediately if φ is a simple function. If $f \in L_1(|\mu^*|)$, then there exists a sequence of simple functions $\{\varphi_i\}_{i=1}^{\infty}$

such that $\varphi_i \to f$ pointwise $|\mu^*|$-a.e. and $|\varphi_i| \leqslant |\varphi_{i+1}| \leqslant |f|$ for all $i \geqslant 1$. Thus the dominated convergence theorem tells us that

$$\int_G f d\mu^* = \lim_{i\to\infty} \varphi_i d\mu^* = \lim_{i\to\infty} \overline{\int_G \overline{\varphi_i \circ \theta} d\mu}.$$

Moreover, since $|T^*| = |T|^*$ for all $T \in C_0^*(G)$, we have that $|\mu^*| = |\mu|^*$ and the monotone convergence theorem tells us that

$$\begin{aligned} \int_G |f \circ \theta| d|\mu| &= \lim_{i\to\infty} \int_G |\varphi_i \circ \theta| d|\mu| = \lim_{i\to\infty} \int_G |\varphi_i| d|\mu|^* \\ &= \lim_{i\to\infty} \int_G |\varphi_i| d|\mu^*| \leqslant \int_G |f| d|\mu^*| < \infty. \end{aligned}$$

Thus $f \circ \theta \in L_1(|\mu|)$ and the dominated convergence theorem tells us that

$$\lim_{i\to\infty} \overline{\int_G \overline{\varphi_i \circ \theta} d\mu} = \overline{\int_G \overline{f \circ \theta} d\mu}.$$

Thus we have shown that if $f \in L_1(|\mu^*|)$ then $f \circ \theta \in L_1(|\mu|)$ and

$$\int_G f d\mu^* = \overline{\int_G \overline{f \circ \theta} d\mu}.$$

The converse follows from the fact that $T^{**} = T^*$ for all $T \in C_0^*(G)$ and therefore $\mu^{**} = \mu$, so that in particular if $f \circ \theta \in L_1(|\mu|) = L_1(|\mu^* *|)$, then $f = f \circ \theta \circ \theta \in L_1(|\mu^*|)$.

The remainder of the theorem then follows immediately from Theorem (50). □

CHAPTER 2. REPRESENTATIONS OF LOCALLY COMPACT GROUPS

2.1. Introduction

The purpose of this chapter is to lay down the necessary theory for representations of a locally compact topological group G. In particular, we are interested in defining representations of groups, the continuity and measureability of representations, and establishing an important result attributed to Gel'fand and Raĭkov that shows that the collection of irreducible unitary representations of a locally compact topological group separates points in G. Throughout this chapter, G will denote a locally compact topological group, X will denote a Banach space, and by H we will mean a Hilbert space[6]. Additionally, by $B(X,X)$ and $U(H)$ we will mean the collection of all bounded operators on X and all unitary operators on H, respectively. We also use X^* to denote the dual space of X. As in the previous chapter, λ will denote a left Haar measure on G and we will use dx and $d\lambda(x)$ interchangeably[7]. In addition, we will use δ_a to denote the dirac measure at $a \in G$ and e to denote the identity of G.

2.2. Basic Definitions and Facts

We begin with a rather large collection of definitions that will setup the framework for the results in this chapter.

Definition 52. We define a *representation of a semigroup S* as homomorphism $\pi : S \to \mathrm{Hom}_k(V,V)$ of S into the semigroup of all linear operators on V, where V is a vector space over a field k. We define a *representation of an algebra A* in the same way, only now π is required to be an algebra homomorphism of A in to the algebra

[6]This notation is different from the previous chapter, where X was a locally compact Hausdorff space and G was used in place of X to represent a locally compact T_0 topological group.

[7]If G is compact, λ will always be the unique Haar measure such that $\lambda(G) = 1$.

$\mathrm{Hom}_k(V, V)$ of linear operators on V. The vector space V is called the representation space of the representation π. A subspace $U \subset V$ is invariant under π if it is invariant under all the linear operators in the image of π: $\pi(a)(U) \subset U$ for all $a \in A$ (or S).

Unless otherwise mentioned, by algebra we will always mean an algebra over $\mathbb{C}$. Moreover, for $k = \mathbb{C}$ we will denote $\mathrm{Hom}_k(V, V)$ as $\mathrm{Hom}(V, V)$. Moreover G will always refer to a locally compact group.

Definition 53. Let V be a topological vector space over a field k, and let $T \in \mathrm{Hom}(V, V)$. A subspace $U \subset V$ is *invariant* under T if $T(U) \subset U$. An artibrary collection $\{T_\alpha\}_{\alpha \in \Lambda} \subset V$ is called *reducible* if there is a nontrivial closed proper subspace invariant under T_α for every $\alpha \in \Lambda$ and *irreducible* otherwise. A representation is called *irreducible* if $\{\pi(x)\}_{x \in G}$ is irreducible.

In our exposition of unitary operators on compact groups, the algebra $\mathbf{M}(G)$ and the concept of the adjoint will play a role. The $*$-algebras that will be of interest to us are those which are subalgebras of $\mathbf{M}(G)$ closed under the adjoint operation $\mu \mapsto \mu^*$. Corresponding to $*$-algebras are $*$-representations.

Definition 54. A $*$-*representation* of a $*$-algebra A is a representation π of A by bounded operators on a Hilbert space H satisfying $\pi(a^*) = \pi(a)^*$ for all $a \in A$. A $*$-representation π is *faithful* if $\pi(a) \neq 0$ for $a \neq 0$.

Definition 55. A representation π of an algebra A or a semigroup S over a Hilbert space H is called *unitary* if $\pi(x)$ is unitary for all $x \in G$. If π_1 and π_2 are representations by bounded operators on Hilbert spaces H_1 and H_2 respectively, then π_1 and π_2 are *equivalent* if there exists a linear isometry $T : H_1 \to H_2$ such that the diagram

$$\begin{array}{ccc} H_1 & \xrightarrow{\pi_1(a)} & H_1 \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ H_2 & \xrightarrow{\pi_2(a)} & H_2, \end{array}$$

commutes for all $a \in A$ (or S). That is, $T\pi_1(a) = \pi_2(a)T$ for all $a \in A$ (or S). This defines an equivalence relation on G.

Definition 56. Let π be a representation of an algebra A (or semigroup S) by operators on a vector space V. Then π defines a (left) action of A (or S) on V. For any $v \in V$, we define the *orbit* v *with respect to* π as $\mathrm{Orb}(v, \pi) = \{\pi(a)v : a \in A\}$ (or $\{\pi(a)v : a \in S\}$). If V is a topological vector space, the representation π is said to be *cyclic* if there exists $v \in V$ such that the subspace of V generated by $\mathrm{Orb}(v, \pi)$ is dense in V. An element v satisfying this property is said to be a *cyclic vector* of π.

Given a representation π of a group G (or algebra A) by operators on a Banach space X there will in general be several induced maps that will have properties necessary in considering questions of continuity and measureability. As a notational convenience, if F is a function whose domain is a subset of $B(X, X)$, the collection of bounded operators on X we will denote the induced mapping with domain G by $F(\pi(\cdot))$. To give a better idea of what we mean, we list a few examples that will recur frequently:

$$\pi(\cdot) : G \to B(X, X) \qquad \text{where } x \mapsto (\pi(\cdot))(x) = \pi(x); \tag{18}$$

$$\pi(\cdot)\psi : G \to X \qquad \text{where } x \mapsto (\pi(\cdot)\psi)(x) = \pi(x)\psi \tag{19}$$

$$\langle \pi(\cdot)\psi, \phi\rangle : G \to \mathbb{C} \qquad \text{where } x \mapsto (\langle \pi(\cdot)\psi, \phi\rangle)(x) = \langle \pi(x)\psi, \phi\rangle, \tag{20}$$

where $\psi \in X$ and $\phi \in X^*$. Using this notation, we establish our next collection of definitions.

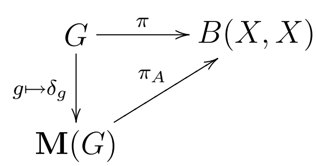
Definition 57. Let X be a Banach space and π be a representation of G by bounded operators on X. The representation π is *weakly Borel measureable* if the function $\langle \pi(\cdot)\psi, \phi\rangle$ is weakly Borel measureable for all $\psi \in X$ and $\phi \in X^*$. The representation is *weakly continuous* if the function $\langle \pi(\cdot)\psi, \phi\rangle$ is continuous for all $\psi \in X$ and $\phi \in X^*$.

If the mapping $\pi(\cdot)\psi$ is continuous for all $\psi \in X$, the representation is called *strongly continuous*. If $\sup_{x \in G} \|\pi(x)\| = c < \infty$, then π is called *totally bounded*.

The motivation for the definition of a totally bounded representation stems from the fact that such representations map G into totally bounded subsets of $B(X, X)$.

2.3. Extensions of Representations to $\mathbf{M}(G)$

Take a representation π of G over a reflexive Banach space X by members of $B(X, X)$[8]. Then π immediately defines a representation of the subalgebra of $\mathbf{M}_d(G)$ in the sense that π maps the atom δ_x to $\pi(x)$. In fact, if π has nice enough properties, we can extend π to a corresponding representation π_A of $\mathbf{M}(G)$ such that the diagram



commutes. The properties required for π to be extended are stated in the next theorem.

Theorem 58. Let A be a subalgebra of $\mathbf{M}(G)$. Let π be a representation of G by bounded operators on a reflexive Banach space X such that

(a) π is weakly $|\mu|$-measureable, and weakly $|\mu| * |\nu|$-measureable for all $\mu, \nu \in A$;

(b) π is totally bounded and $\sup_{x \in G} \|\pi(x)\| = c$.

Then for every $\mu \in A$, there exists a unique operator $\pi_A(\mu)$ such that

$$\langle \pi_A(\mu) f, h \rangle = \int_G \langle \pi(x) f, h \rangle d\mu(x) \tag{21}$$

[8] A reflexive Banach space is a Banach space X where the dual space of the dual space X^{**} is isometrically isomorphic to X. In this case, we use the convention $X = X^{**}$ and will let $f \in X$ denote both itself and it's counterpart in X^{**} under the isometry.

for all $f \in B$ and $h \in B^*$. Moreover the mapping $\mu \mapsto \pi_A(\mu)$ is a representation of A by bounded operators on B such that $\|\pi_A(\mu)\| \leqslant c\|\mu\|$ for all $\mu \in A$.

Proof. Let $\mu \in A$. For any $\psi \in X$ and $\phi \in X^*$, since $\langle \pi(x)\psi, \phi\rangle$ is $|\mu|$-measureable and $|\langle \pi(x)\psi, \phi\rangle| \leqslant \|\pi(x)\psi\|\|\phi\| \leqslant c\|\psi\|\|\phi\|$, we know that $\langle \pi(x)\psi, \phi\rangle \in L_1(\mu)$. Moreover for fixed $\psi \in X$ and $x \in G$, the mapping

$$h \mapsto \int_G \langle \pi(x)\alpha\psi, \phi\rangle d\mu(x)$$

is obviously conjugate-linear and

$$\left|\int_G \langle \pi(x)\alpha\psi, \phi\rangle d\mu(x)\right| \leqslant \int_G |\langle \pi(x)\alpha\psi, \phi\rangle| d|\mu|(x) \leqslant c\|\psi\|\|\phi\||\mu|(G) < \infty$$

so that it is necessarily bounded. Every bounded conjugate-linear functional on the reflexive Banach space X^* is of the form $\phi \mapsto \langle \widetilde{\psi}, h\rangle$ for some $\widetilde{\psi} \in X^{**} = X$. Identifying $\widetilde{\psi}^{**}$ with $\pi_A(\mu)\psi$ in the bounded conjugate-linear functional above defines a mapping $\pi_A(\mu)$ on X satisfying Equation (21). Let $\psi_1, \psi_2 \in X$. For all $\phi \in X^*$, we have that

$$\begin{aligned}
\langle \pi_A(\mu)(\psi_1 + \psi_2), h\rangle &= \int_G \langle \pi(x)(\psi_1 + \psi_2), \phi\rangle d\mu(x) \\
&= \int_G \langle \pi(x)\psi_1, \phi\rangle d\mu(x) + \int_G \langle \pi(x)\psi_2, \phi\rangle d\mu(x) \\
&= \langle \pi_A(\mu)\psi_1, \phi\rangle + \langle \pi_A(\mu)\psi_1, \phi\rangle
\end{aligned}$$

which implies $\pi_A(\mu)(\psi_1 + \psi_2) = \pi_A(\mu)\psi_1 + \pi_A(\mu)\psi_2$. Similarly $\pi_A(\mu)(\alpha\psi) = \alpha\pi_A(\mu)\psi$ for all $\psi \in X$ and $\alpha \in \mathbb{C}$, so that $\pi_A(\mu)$ is a linear transformation on X. The inequality $|\langle \pi(\mu)\psi, \phi\rangle| \leqslant c\|\psi\|\|\phi\||\mu|(G)$ tells us

$$\|\pi_A(\mu)\| = \sup_{0\neq\psi\in B} \frac{\|\pi_A(\mu)\psi\|}{\|\psi\|} = \sup_{0\neq\phi\in X^*} \frac{|\langle \pi_A(\mu)\psi, \phi\rangle|}{\|\psi\|\|\phi\|} \leqslant c|\mu|(G) = c\|\mu\|.$$

At this point, we have shows that $\mu \mapsto \pi_A(\mu)$ is a function from A into $B(X, X)$. The linearity relations $\pi_A(\mu + \nu) = \pi_A(\mu) + \pi_A(\nu)$ and $\pi_A(\alpha\mu) = \alpha\pi_A(\mu)$ for all $\mu, \nu \in A$ and $\alpha \in \mathbb{C}$ are immediate consequences of the definition of of π_A. Thus we need only verify the relation $\pi_A(\mu * \nu) = \pi_A(\mu)\pi_A(\nu)$ for all $\mu, \nu \in A$. For any $\psi \in X$ and $\phi \in X^*$, we have that

$$\begin{aligned}
\langle \pi_A(\mu * \nu)\psi, \phi\rangle &= \int_G \langle \pi(x)\psi, \phi\rangle d(\mu * \nu)(x) = \int_G \int_G \langle \pi(xy)\psi, h\rangle d\nu(y) d\mu(x) \\
&= \int_G \int_G \langle \pi(x)\pi(y)\psi, \phi\rangle d\nu(y) d\mu(x) \\
&= \int_G \int_G \langle \pi(y)\psi, (\pi(x))^*\phi\rangle d\nu(y) d\mu(x) = \int_G \langle \pi_A(\nu)\psi, (\pi(x))^*\phi\rangle d\mu(x) \\
&= \int_G \langle \pi(x)\pi_A(\nu)\psi, \phi\rangle d\mu(x) = \langle \pi_A(\mu)\pi_A(\nu)\psi, \phi\rangle.
\end{aligned}$$

This proves our theorem. □

In the above, we considered a subalgebra A of $\mathbf{M}(G)$. Clearly if A contains $\mathbf{M}_d(G)$, then $\pi(\delta_g) = \pi(g)$. To establish the requirements for the representation to be a $*$-representation, we first consider the "adjoint algebra" $A^* = \{\mu^* : \mu \in A\}$. For any subalgebra A of $\mathbf{M}(G)$, A^* is also a subalgebra of A, and a representation π extendable to a representation of A is also extendable to a representation of A^*. Moreover, if A is closed under adjoints, B is a Hilbert space, and the representation π is unitary, then the extension π_A is a $*$-representation of $\mathbf{M}(G)$. This is made clear by the following theorem.

Theorem 59. Let A be a subalgebra of $\mathbf{M}(G)$ and π a representation of A by bounded operators on a reflexive Banach space X satisfying the assumptions of Theorem (58).

Then for every $\mu^* \in A^*$, there exists a unique operator $\widetilde{\pi_A}(\mu)$ such that

$$\langle \widetilde{\pi_A}(\mu^*)\phi, \psi\rangle = \int_G \langle (\pi(x^{-1}))^*\phi, \psi\rangle d\mu^*(x) = \langle \phi, \pi_A(\mu)\psi\rangle \tag{22}$$

for all $\phi \in X^*$ and $\psi \in X^{**} = X$. For every $\mu \in A$, we have $\widetilde{\pi_A}(\mu^*) = (\pi_A(\mu))^*$. If X is a Hilbert space and we identify $X = X^* = H$, and π is unitary, then $\pi_A(\mu) = \widetilde{\pi_A}(\mu)$, so that π is a $*$-representation of the $*$-algebra A.

Proof. Define a function $\widetilde{\pi}$ mapping G to operators on X^* via $\widetilde{\pi} : \mathbf{x} \mapsto (\pi(x^{-1}))^*$. Since π is a representation of G by bounded operators on X, we see that

$$(\pi((xy)^{-1}))^* = (\pi(y^{-1})\pi(x^{-1}))^* = (\pi(x^{-1}))^*(\pi(y^{-1}))^*,$$

and therefore $\widetilde{\pi}$ is a representation of G by bounded operators on X^*. Since

$$\langle \widetilde{\pi}(x)\phi, \psi\rangle = \overline{\langle \psi, (\pi(x^{-1}))^*\phi\rangle} = \overline{\langle (\pi(x^{-1}))\psi, \phi\rangle},$$

property (a) of Theorem (58) holds for $\widetilde{\pi}$. Furthermore, $\widetilde{\pi}(x^{-1}) = (\pi(x))^*$ and $\|(\pi(x))\| = \|(\pi(x))^*\|$, and therefore property (b) of Theorem (58) holds for the representation π^*. The previous theorem then grants us the existence of a representation $\widetilde{\pi_A}$ satisfying the left-hand side equality of Equation (22). The other equality in Equation (22) follows from a standard manipulation:

$$\begin{aligned}\langle \widetilde{\pi_A}(\mu^*)\phi, \psi\rangle &= \int_G \langle (\pi(x^{-1}))^*\phi, \psi\rangle d\mu^*(x) = \int_G \langle \phi, \pi(x^{-1})\psi\rangle d\mu^*(x) \\ &= \int_G \overline{\langle \pi(x^{-1})\psi, \phi\rangle} d\mu^*(x) = \overline{\int_G \langle \pi(x)\psi, \phi\rangle d\mu(x)} = \langle \phi, \pi_A(\mu)\psi\rangle.\end{aligned}$$

In particular, this implies that $(\pi_A(\mu))^* = \widetilde{\pi_A}(\mu^*)$. The last part of the theorem is a consequence of the fact that for unitary operators $\pi(x^{-1}) = (\pi(x))^*$, so that

$\widetilde{\pi}(x^{-1}) = \pi(x)$ and therefore $(\pi_A(\mu))^* = \widetilde{\pi}_A(\mu^*) = \pi_A(\mu)$. □

An important question to try to answer at this point concerns the invariance of sets under the extension we have described above. In particular, we are interested in finding a general set of conditions under which a π invariant subspace of a Hilbert space H is μ_A invariant and visa-versa. As it turns out, if the representation π is unitary, the extension π_A defined by Theorem (58) respects invariance in the sense that subspaces of the corresponding Hilbert space invariant under π are also invariant under π_A. We have the converse when π is weakly continuous and for every open set U of G, A contains a nonzero positive measure supported on U. This is essentially the content of the next theorem.

Theorem 60. Let A be a $*$-subalgebra of $\mathbf{M}(G)$ and π a representation of G by unitary operators on a Hilbert space H satisfying (a) of Theorem (58) (since π is unitary, it automatically satisfies (b)). Then every closed subspace of H invariant under π is invariant under π_A. If π is weakly continuous as a representation of G and for every nonvoid open subset U of G there exists a nonnegative real measure $\mu \in A$ such that $\mu(U) = 1$ and $\mu(U^c) = 0$, then every closed subspace of H invariant under π_A is invariant under π.

Proof. Let V be a closed subspace of H invariant with respect to π and P be the projection of H onto V. Let T be any operator on H. Given any $f \in H$, we may write f as the linear combination $\psi = \psi_1 + \psi_2$ where $\psi_1 = P\psi$ and $\psi_2 = \psi - \psi_1$. Clearly $P\psi_1 = \psi_1$ and $P\psi_2 = 0$. It follows that $TP\psi = T\psi_1$, so that $PT\psi = TP\psi$ if and only if $PT\psi = T\psi_1$. Thus V is invariant with respect to an operator T on H if and only if P commutes with T. It follows that $P\pi(x) = \pi(x)P$ for all $x \in G$. Recall that $P^* = P$. Then for all $\mu \in A$ and $\psi, \phi \in H$, we have that

$$\begin{aligned}\langle \pi_A(\mu) P\psi, \phi\rangle &= \int_G \langle \pi(x) P\psi, \phi\rangle d\mu(x) = \int_G \langle P\pi(x)\psi, \phi\rangle d\mu(x) \\ &= \int_G \langle \pi(x)\psi, P\phi\rangle d\mu(x) = \langle \pi_A(\mu)\psi, P\phi\rangle = \langle P\pi_A(\mu)\psi, \phi\rangle.\end{aligned}$$

Therefore $\pi_A(\mu)P = P\pi_A(\mu)$ for all $\mu \in A$. This proves that V is invariant with respect to π_A.

To prove the second part of the theorem, suppose that V is a closed subspace of H invariant with respect to π_A, and let P be the projection of H onto V. Fix $\psi, \phi \in H$. From a previous calculation, we know that

$$\langle P\pi_A(\mu)\psi, \phi\rangle = \int_G \langle P\pi(x)\psi, \phi\rangle d\mu(x).$$

Since $\pi_A P = P\pi_A$, we have that

$$0 = \langle P\pi_A(\mu)\psi, \phi\rangle - \langle \pi_A P\psi, \phi\rangle = \int_G \langle (P\pi(x) - \pi(x)P)f, h\rangle d\mu(x)$$

for all $\mu \in A$. Since π is weakly continuous, the function $\langle (P\pi(x) - \pi(x)P)\psi, \phi\rangle$ is a continuous bounded function of x. Thus if there exists an $a \in G$ with $\langle (P\pi(a) - \pi(a)P)\psi, \phi\rangle \neq 0$, then there must be a nonempty open subset U of G such that the real or imaginary portion of $\langle (P\pi(a) - \pi(a)P)\psi, \phi\rangle$ does not change sign on U. Taking μ to be the positive measure on U with $\mu(U) = 1$ and $\mu(U^c) = 0$, we find that the integral

$$\int_G \langle (P\pi(x) - \pi(x)P)\psi, \phi\rangle d\mu(x) = \int_U \langle (P\pi(x) - \pi(x)P)\psi, \phi\rangle d\mu(x) \neq 0,$$

which is a contradiction. We conclude that $\langle (P\pi(x) - \pi(x)P)f, h\rangle = 0$ for all $x \in G$.

Since $\psi, \phi \in H$ was arbitrary, this proves that $\pi(x)$ and P commute. Thus V is invariant with respect to π. □

2.4. Continuity of Representations

In this section, we discuss the continuity of representations. Specifically, we obtain the result that absolutely bounded weakly continuous representations are strongly continuous. These representations turn out to have the utility we need, and we simply call them "continuous representations".

Theorem 61. Let π be a weakly λ-measureable representation of G by bounded operators on a reflexive Banach space X that is absolutely bounded, and suppose the function $\langle \pi(\cdot)f, h\rangle$ is continuous at e for all $\psi \in X$ and $\phi \in X^*$. Then the function $\pi(\cdot)\psi$ is a left uniformly continuous mapping of G into X with respect to the norm topology on X. In particular, every weakly continuous totally bounded representation of G is strongly continuous.

Proof. Since π is λ-measureable, π property (a) of Theorem (58) for $A = \mathbf{M}_a(G)$. Extend π to a representation π_A of $\mathbf{M}_a(G)$ as in Equation (21) of Theorem (58). We first show that each element $\psi \in X$ is contained in the closure of its orbit with respect to π_A. This will be done by contradiction. Thus assume that $\psi \in X$ such that $\psi \notin \overline{\mathrm{Orb}(\psi, \pi_A)}$. Then by the Hahn-Banach Theorem (see [6]), there exists a linear functional $\phi \in X^*$ with $\langle \psi, \phi\rangle = 1$ and $\langle \mathrm{Orb}(\psi, \pi_A), \phi\rangle = \{0\}$. Fix a positive constant $0 < c < 1$. Since $\langle \pi(e)\psi, \phi\rangle = 1$ and $\langle (\pi(\cdot)\psi, \phi\rangle$ is continuous at e, there exists a neighborhood U of the identity $e \in G$ with $\lambda(U) > 0$ such that $\mathrm{Re}\langle \pi(\cdot)\psi, \phi\rangle > c$ on U. If we let $\mu \in \mathbf{M}_a(G)$ be the measure such that $d\mu = 1_U d\lambda$, then

$$\mathrm{Re}\langle \pi(\mu)\psi, \phi\rangle = \int_G \mathrm{Re}\langle \pi(x)\psi, \phi\rangle 1_U dx = \int_U \mathrm{Re}\langle \pi(x)\psi, \phi\rangle dx > c\lambda(U) > 0.$$

This is a contradiction. We conclude that $\psi \in \overline{\mathrm{Orb}(\psi, \pi_A)}$ for all $\psi \in X$.

Define $b = \sup_{x\in G} \|\pi(x)\|$. Consider any $a \in G$, $\psi \in X$, $\phi \in X^*$, and $\mu \in \mathbf{M}_a(G)$. A quick calculation shows

$$\begin{aligned}\langle \pi(\delta_a * \mu)\psi, \phi\rangle &= \int_G \langle \pi(x)\psi, \phi\rangle d(\delta_a * \mu)(x) = \int_G\int_G \langle \pi(xy)\psi, \phi\rangle \delta_a(x) d\mu(y) \\ &= \int_G \langle \pi(ay)\psi, \phi\rangle d\mu(y) = \int_G \langle \pi(a)\pi(y)\psi, \phi\rangle d\mu(y) \\ &= \int_G \langle \pi(y)\psi, \pi(a)^*\phi\rangle d\mu(y) = \langle \pi_A(\mu)\psi, \pi(a)^*\phi\rangle = \langle \pi(a)\pi_A(\mu)\psi, \phi\rangle.\end{aligned}$$

In particular,

$$|\langle \pi(a)\pi_A(\mu)\psi - \pi_A(\mu)\psi, \psi\rangle| \leqslant |\langle (\pi_A(\delta_a * \mu) - \pi_A(\mu))\psi, \phi\rangle| \leqslant b\|\psi\|\|\phi\|\delta_a * \mu - \mu\|.$$

Since $\mu \ll \lambda$, $d(\delta_a * \mu) = L(a)\frac{d\mu}{d\lambda}d\lambda$. Since $x \mapsto L(x)\frac{d\mu}{d\lambda}$ is a right uniformly continuous function from G into $L_1(G)$, for all $\epsilon > 0$, there exists a neighborhood U_μ of e in G such that

$$\begin{aligned}\|\delta_a * \mu - \mu\| = |\delta_a * \mu - \mu|(G) &= \int_G \left|\left(L(a)\frac{d\mu}{d\lambda}\right)(x) - \frac{d\mu}{d\lambda}(x)\right| dx \\ &= \left\|L(a)\frac{d\mu}{d\lambda} - \frac{d\mu}{d\lambda}\right\|_1 < \frac{\epsilon}{2\|\psi\|b^2}\end{aligned}$$

for all $a \in U_\mu$. Therefore

$$\|\pi(a)\pi_A(\mu)\psi - \pi_A(\mu)\psi\| \leqslant \frac{\epsilon}{2b}$$

for all $a \in U_\mu$. Since ψ is in the closure of the $\mathrm{Orb}(\psi, \pi_A)$, we may choose $\mu \in \mathbf{M}_a(G)$ such that

$$(1 + b)\|\psi - \pi_A(\mu)\psi\| < \frac{\epsilon}{2b}.$$

Then we have that for all $x \in U_\mu$

$$\begin{aligned}\|\pi(x)\psi - \psi\| &\leqslant \|\pi(x)\psi - \pi(x)\pi_A(\mu)\psi\| + \|\pi(x)\pi_A(\mu)\psi - \pi_A(\mu)\psi\| + \|\pi_A(\mu)\psi - \psi\| \\ &\leqslant b\|\psi - \pi_A(\mu)\psi\| + \frac{\epsilon}{2b} + \|\pi_A(\mu)\psi - \psi\| < \frac{\epsilon}{2b} + \frac{\epsilon}{2b} = \frac{\epsilon}{b}.\end{aligned}$$

Thus, if $x, y \in G$ with $y^{-1}x \in U_\mu$, we find

$$\|\pi(x)\psi - \pi(y)\psi\| = \|\pi(y)(\pi(y^{-1}x)\psi - \psi)\| \leqslant \|\pi(y)\|\|\pi(y^{-1}x)\psi - \psi\| < \epsilon.$$

This proves our theorem. □

In light of this theorem, we make the following definition.

Definition 62. Let π be a representation of G by operators on a reflexive Banach space X under which the mapping $\pi(\cdot)\psi$ is a left uniformly continuous mapping of G onto X with respect to the topology defined by the norm of X. Then π is called a *continuous representation* of G. In particular, weakly λ-measureable, weakly continuous or strongly continuous, and absolutely bounded representations are continuous representations of (G).

2.5. Restriction of Representations of $\mathbf{M}(G)$

Given a representation π of G, we have found some general conditions under which we may extend to a representation π_A of a $*$-subalgebra A of $\mathbf{M}(G)$. The use of "extend" entails that we are considering G as a subset of A, via the correspondence $g \mapsto \delta_g$. To be more accurate, π_A is an extension of a representation of the $*$-subalgebra of A generated by the atoms δ_g. There is an obvious problem with this: the $*$-algebra A may or may not contain the point masses. We have an alternative when A is closed under the operation of convolution with point masses and the representation π_A satisfies certain nice properties. This is the topic of the next theorem.

Theorem 63. Let A be a Banach $*$-subalgebra of $\mathbf{M}(G)$ such that $\delta_g * \mu \in A$ for all $g \in G$ and $\mu \in A$, and let π_A be a $*$-representation of A by bounded operators on a Hilbert space H such that for all $\psi \in H$ there exists $\mu \in \mathbf{M}(G)$ such that $\pi_A(\mu)\psi \neq 0$. Then there exists a representation π of G by unitary operators on H.

Proof. We can decompose the Hilbert space H in to a direct sum $H = N \oplus \bigoplus_{\gamma\in\Gamma} H_\gamma$ of π_A-invariant subspaces of H, where $\pi_A(\mu)(N) = \{0\}$ for all $\mu \in \Gamma$ and the restriction of the representation π_A to H_γ is cyclic. By assumption $N = \{0\}$. Let P_γ be the projection operator of H onto H_γ. Suppose that for each γ we have a unitary representation π_γ of G by unitary operators on H_γ satisfying the relation

$$\langle \pi_A(\mu) P_\gamma \psi, P_\gamma \phi \rangle = \int_G \langle \pi_\gamma(x) P_\gamma \psi, P_\gamma \phi \rangle d\mu(x)$$

for all $\psi, \phi \in H$. For any $a \in G$, define $\pi(a)$ to be the operator on H defined by

$$\pi(a)\psi = \bigoplus_{\gamma\in\Gamma} \pi_\gamma(a) P_\gamma.$$

This is a direct sum of mutually orthogonal unitary operators, and therefore defines a unitary operator on H. Moreover this orthogonality gives us the identity

$$\pi(xy) = \bigoplus_{\gamma\in\Gamma} \pi_\gamma(xy) P_\gamma = \bigoplus_{\gamma\in\Gamma} \pi_\gamma(x)\pi_\gamma(y) P_\gamma = \bigoplus_{\gamma,\gamma'\in\Gamma} \pi_\gamma(x) P_\gamma \pi_{\gamma'}(y) P_\gamma = \pi(x)\pi(y),$$

and therefore π is a unitary representation of G. From its definition, π satisfies Equation (21).

Thus we need only prove our theorem for the special case that π_A is a cyclic representation. With this in mind, we assume that π_A is a cyclic $*$-representation of A satisfying the properties of the theorem. Then there exists a vector $\varphi \in H$ such that $\mathrm{Orb}(\varphi, \pi_A)$ is dense in H. Using the fact that $\delta_x^* = \delta_{x^{-1}}$ and π_A is a $*$-algebra representation, we have the relation $[\pi_A(\delta_x * \mu)]^* = \pi_A(\mu^* * \delta_x^*) = \pi_A(\mu^* * \delta_{x^{-1}})$ for all $\mu \in A$. Thus

$$\begin{aligned}\|\pi_A(\delta_x * \mu)\varphi\|^2 &= \langle \pi_A(\delta_x * \mu)\varphi, \pi_A(\delta_x * \mu)\varphi\rangle = \langle f, [\pi_A(\delta_x * \mu)]^* \pi_A(\delta_x * \mu)\varphi\rangle \\ &= \langle \varphi, \pi_A(\mu^* * \delta_{x^{-1}})\pi_A(\delta_x * \mu)\varphi\rangle = \langle \varphi, \pi_A(\mu^* * \delta_{x^{-1}} * \delta_x * \mu)\varphi\rangle \\ &= \langle \varphi, \pi_A(\mu^* * \mu)\varphi\rangle = \langle \varphi, [\pi_A(\mu)]^* \pi_A(\mu)\varphi\rangle = \langle \pi_A(\mu)\varphi, \pi_A(\mu)\varphi\rangle \\ &= \|\pi_A(\mu)\varphi\|^2\end{aligned}$$

for all $\mu \in A$ and $x \in G$. For every $x \in G$, define an operator $\pi'(x)$ on $\mathrm{Orb}(\varphi, \pi_A)$ by $\pi'(x)(\pi_A(\mu)\varphi) = \pi_A(\delta_x * \mu)$. If $\mu, \nu \in A$ satisfy $\pi_A(\mu)\varphi = \pi_A(\nu)\varphi$, then the above calculation shows that

$$\begin{aligned}\|\pi'(x)(\pi_A(\mu)\varphi) - \pi'(x)(\pi_A(\nu)\varphi)\| = \|\pi_A(\delta_x * \mu)\varphi - \pi_A(\delta_x * \nu)\varphi\| = \\ \|\pi_A(\delta_x * (\mu - \nu))\varphi\| = \|\pi_A(\mu - \nu)\varphi\| = \|\pi_A(\mu)\varphi - \pi_A(\nu)\varphi\| = 0.\end{aligned}$$

Thus the mapping $\pi'(x)$ is well-defined. Moreover, for any $\mu, \nu \in A$ and $\alpha \in \mathbb{C}$, we have that

$$\begin{aligned}\pi'(x)(\pi_A(\mu)\varphi + \pi_A(\nu)\varphi) = \pi'(x)(\pi_A(\mu + \nu)\varphi) = \pi_A(\delta_x * (\mu + \nu))\varphi \\ = \pi_A(\delta_x * \mu + \delta_x * \nu)\varphi = \pi_A(\delta_x * \mu)\varphi + \pi_A(\delta_x * \nu)\varphi = \pi'(x)(\pi_A(\mu)\varphi) + \pi'(\pi_A(\nu)\varphi)\end{aligned}$$

and similarly $\pi'(x)(\pi_A(\alpha\mu)\varphi) = \alpha\pi'(x)(\pi_A(\mu)\varphi)$. Moreover, we have already shown that $\pi'(x)$ is norm-preserving. We therefore conclude that $\pi'(x)$ is a linear isometry on $\mathrm{Orb}(\varphi, \pi_A)$ for all $x \in G$. Since $\mathrm{Orb}(\varphi, \pi_A)$ is dense in H, this representation $\pi'(x)$ can be extended to a unique linear isometry (and therefore unitary operator) $\pi(x)$ on H. Since $\delta_{xy} = \delta_x * \delta_y$, it is evident that $\pi'(xy) = \pi'(x)\pi'(y)$, from which it immediately follows that $\pi(xy) = \pi(x)\pi(y)$. Hence π is a representation of G by unitary operators on H. This proves our theorem. $\square$

The generality of the conditions under which we can restrict a representation π_A of A to a representation π of G is good news. However, without knowing more about the properties of the representation π_A it would be difficult to discuss the properties of π. In particular, we will not know in general if π is continuous, absolutely bounded, or even measureable. In the next theorem, we consider some conditions under which a representation of a $*$-subalgebra A of $\mathbf{M}(G)$ can be pulled back to a representation of G with certain nice properties.

Theorem 64. Let A be a Banach $*$-subalgebra of $\mathbf{M}(G)$ and π_A a representation of A by bounded operators on a Hilbert space H satisfying the assumptions of Theorem (63). Additionally suppose every bounded linear functional on A has the form $\mu \mapsto \int_G l(x)d\mu(x)$ for some bounded function l on G such that for every σ-compact subset $S \subset G$, $1_S l$ is Borel measureable. Then there is a representation π of G by unitary operators on H that is weakly μ and $\mu * \nu$-measureable for all $\mu, \nu \in A$ satisfying Equation (21).

Proof. We have already shown the existence of a unitary representation π in Theorem (63), so we need only consider the measureability condition and also show that this representation satisfies Equation (21). Using the notation of the previous theorem, to show that π is weakly $|\mu|$-measureable, note that $\langle \pi(g)f, h\rangle = \bigoplus_{\gamma\in\Gamma} \langle \pi_\gamma(g)P_\gamma f, h\rangle$. Thus if each of the π_γ is weakly $|\mu|$-measureable, $\langle \pi(g)f, h\rangle$ is a countable sum of $|\mu|$-measureable functions and therefore measureable. It follows that π is $|\mu|$-measureable. Thus we need only consider the special case when H is cyclic, with cyclic vector φ. Let $h \in H$. Any $*$-representation π_A is bounded (see [2], pp. 320), so the mapping $\mu \mapsto \langle \rho(\mu)\varphi, h\rangle$ is a bounded linear functional on A for every $h \in H$. Thus by assumption, there exists a function l such that

$$\langle \rho(\mu)\varphi, \phi\rangle = \int_G l(x)d\mu(x)$$

for all $\mu \in A$. Moreover

$$\begin{aligned}\langle \pi(g)\pi_A(\mu)\varphi, h\rangle = \langle \pi_A(\delta_g * \mu)\varphi, h\rangle &= \int_G l(x)d(\delta_g * \mu)(x) \\ &= \int_G \int_G l(xy)d\delta_g(x)d\mu(y) = \int_G l(gy)d\mu(y).\end{aligned}$$

Let μ and ν be any measures on $\mathbf{M}(G)$. Since μ and ν are finite, we may choose σ-compact subsets $S_\mu, S_\nu \subset G$ such that $|\mu|$ is supported on S_μ and $|\nu|$ is supported on S_ν. It follows that $|\mu| \times |\nu|((S_\mu \times S_\nu)^c) = 0$. Let $\tau : G \times G \to G$ be the binary operation defined by $\tau(x, y) = xy$. Since $S_\mu \times S_\nu \subset \tau^{-1}(S_\mu S_\nu)$, the identity $l(xy) = (h \circ \tau)(x, y) = (l \circ \tau)(x, y)1_{S_\mu S_\nu}(x, y) = l(xy)1_{S_\mu S_\nu}$ holds $|\mu| \times |\nu|$-a.e.. Since $l1_{S_\mu S_\nu}$ is Borel measureable by assumption, we have that $l \circ \tau \in L_1(|\mu| \times |\nu|)$ and therefore the function

$$x \mapsto \int_G l(xy)d\nu(y) = \langle \pi(x)\pi_A(\nu)\varphi, \phi\rangle$$

is $|\mu|$-measureable for all $\mu \in \mathbf{M}(G)$. Since $\mathrm{Orb}(\varphi, \pi_A)$ is dense in H, we have that $x \mapsto \langle \pi(x)\psi, \phi\rangle$ is $|\mu|$-measureable for all $\mu \in \mathbf{M}(G)$ and $\psi \in H$.

A quick calculation now shows that

$$\begin{aligned}\int_G \langle \pi(x)\pi_A(\nu)\varphi, \phi\rangle d\mu(x) = \int_G \int_G l(xy)d\nu(y)d\mu(x) &= \int_G l(z)d(\mu * \nu)(z) \\ = \langle \pi_A(\mu * \nu)\varphi, \phi\rangle &= \langle \pi_A(\mu)\pi_A(\nu)\varphi, \phi\rangle.\end{aligned}$$

Since the collection of elements of the form $\pi_A(\nu)\varphi$ is dense in H, the equality

$$\int_G \langle \pi(x)\psi, \phi\rangle d\mu(x) = \langle \pi_A(\mu)\psi, \phi\rangle$$

holds in general. Since $\phi \in H$ was arbitrary, this proves our theorem. □

The theorem above can be applied specifically to the subalgebra $\mathbf{M}_a(G)$ of G. In doing so, the representation that we end up with is in fact a continuous representation. Before we prove this, we must establish a technical lemma.

Lemma 65. Let h be a real or complex-valued function on G. Then there exists a λ-measureable fuction $\widetilde{h}$ on G equal to h locally λ-a.e., and $\widetilde{h}1_B$ is Borel measureable for all σ-compact subsets $B \subset G$.

Proof. We prove this for the case that h is real-valued and positive, and the more general case follows immediately. Let U be a conditionally compact symmetric neighborhood of e in G, and set $L = \bigcup_{i=1}^{\infty} U^i$[9]. Then if $x, y \in L$, $x \in U^j$ and $y \in U^k$ for some integers $j, k > 0$. It follows that $xy \in U^{j+k} \subset L$, and therefore L is a subgroup of G. The set L is open, since it is a union of open sets. Since L is a subgroup, $L^c = \bigcup_{g \in G \setminus L} gL$. Since translation is a continuous operation, gL is open for every $g \in G$. We conclude that L^c is also a union of open sets, and therefore L is closed. By its construction, L is σ-compact.

Let S be a subset of G such that $\{sL\}_{s\in S}$ forms the collection of all distinct left cosets of L in G, and define $h_s = h1_{sL}$ for all $s \in S$. For each $s \in S$, sL is σ-compact, and therefore there exists a monotone increasing sequence of compact sets $\{F_i^{(s)}\}_{i=1}^{\infty}$ such that $sL = \bigcup_{i=1}^{\infty} F_i^{(s)}$. For every positive integer n, define $g_n^{(s)}$ to be the function

[9] An open set U is called conditionally compact if its closure is compact. It is called symmetric if $x^{-1} \in U$ for every $x \in U$.

supported on $F_n^{(s)}$ such that

$$g_n^{(s)}(x) = \begin{cases} h_s(x) & h_s(x) \leqslant n; \\ n & \text{otherwise} \end{cases},$$

for all $x \in F_n^{(s)}$. Then $g_n^{(s)} \uparrow h_s$ and each $g_n^{(s)}$ is bounded with compact support, and therefore is equal to a λ-measureable function $\widetilde{g}_n^{(s)}$ λ-a.e.. It follows that $\widetilde{g}_n^{(s)} \to \widetilde{h}_s$, where $\widetilde{h}_s$ is a λ-measureable function and $\widetilde{h}_s = h_s$ λ-a.e.. The cosets $\{sL\}_{s\in S}$ are mutually disjoint, and their union is G, so the function $\widetilde{h}$ defined by $\widetilde{h}(x) = h_s(x)$ for $x \in sL$ is a well-defined function on G. By its definition, $\widetilde{h} = h$ locally λ-a.e.. If $B \subset G$ is σ-compact, there exists a countable subcover $\{s_iL\}_{i=1}^{\infty}$ of the open cover $\{sL\}_{s\in S}$ of B, from which it follows $\widetilde{h}1_B = 1_B \sum_{i=1}^{\infty} h_{s_i}$. This is the product of Borel measureable functions, and therefore Borel measureable. This proves our lemma. $\square$

Theorem 66. Let π_A be a $*$-representation of the algebra $\mathbf{M}_a(G)$ by operators on a Hilbert space H such that for all nonzero $f \in H$ there exists a $\mu \in \mathbf{M}_a(G)$ such that $\rho(\mu)f \neq 0$. Then there exists a continuous unitary representation π of G by unitary operators on H satisfying Equation (21) for all $\mu \in \mathbf{M}_a(G)$ and $f, h \in H$. Moreover, this representation is unique in the sense that it is the only continuous unitary representation of G satisfying Equation (21).

Proof. We know that $\mathbf{M}_a(G)$ is a subalgebra of $\mathbf{M}(G)$ and that every bounded linear functional on $\mathbf{M}_a(G)$ has the form

$$\mu \mapsto \int_G h(x) d\mu(x)$$

for some $h \in L_\infty(G)$. By Lemma (65), we may assume that $1_S h$ is Borel measureable for all σ-compact subsets $S \subset G$. It follows that $\mathbf{M}_a(G)$ satisfies the assumptions of Theorem (64), so we have a unitary representation π of G satisfying Equation (21) that is weakly μ-measureable for all $\mu \in \mathbf{M}(G)$. We now wish to show that π is continuous. As in the proof of Theorem (63) H can be decomposed into the direct sum $H = \bigoplus_{\gamma\in\Gamma} H_\gamma$ of π_A-invariant subspace of H, where the restriction of π_A to H_γ is cyclic for all $\gamma \in \Gamma$. For each $\gamma \in \Gamma$, let φ_γ be a cyclic vector of the restriction of π_A to H_γ. Then as in the proof of Theorem (63), for any $\mu \in \mathbf{M}_a(G)$

$$\begin{aligned}\|\pi(x)\pi_A(\mu)\varphi_\gamma - \pi_A(\mu)\varphi_\gamma\| &= \|\pi_A(\delta_x * \mu)\varphi_\gamma - \pi_A(\mu)\varphi_\gamma\| \\ &\leqslant \|\delta_x * \mu - \mu\|\|\varphi_\gamma\| = \left\| L(x)\frac{d\mu}{d\lambda} - \frac{d\mu}{d\lambda}\right\|_1 \|\varphi_\gamma\|.\end{aligned}$$

The mapping $L(\cdot)f_\mu : x \mapsto L(x)\frac{d\mu}{d\lambda}$ is right uniformly continuous, so this shows us that the mapping $\pi(\cdot)\psi : x \mapsto \pi(x)\psi$ is left uniformly continuous for all $\psi \in \mathrm{Orb}(\varphi_\gamma, \pi_A)$. Since $\mathrm{Orb}(\varphi_\gamma, \pi_A)$ is dense in H_γ, this proves left uniform continuity for all $\psi \in H_\gamma$. The uniform continuity of $\pi(\cdot)\psi$ for all $\psi \in H$ follows immediately. Thus π is continuous.

If ρ is another continuous representation of G by unitary operators on H satisfying Equation (21), then

$$0 = \langle \pi_A(\mu)\psi, \phi\rangle - \langle \pi_A(\mu)\psi, \phi\rangle = \int_G \langle (\pi(x) - \rho(x))\psi, \phi\rangle d\mu(x)$$

for all $\mu \in \mathbf{M}_a(G)$ and $\psi, \phi \in H$. In particular, if $\langle (\pi(x) - \rho(x))\psi, \phi\rangle \neq 0$ for some $x \in G$, then by continuity we know that there exists a set of nonzero positive λ measure $S \subset G$ such that the real or imaginary part of $\langle (\pi(x) - \rho(x))\psi, \phi\rangle$ does not change sign on S. This leads immedately to a contradiction by using $d\mu = 1_S dx$ in the integral above. This proves our theorem. □

2.6. The Regular Representation of $\mathbf{M}(G)$ and Gel'fand-Raĭkov

Our most important example of a $*$-representation of $\mathbf{M}(G)$ will be the "regular representation”, which is a faithful representation by operators on the Hilbert space $L_2(G)$.

Definition 67. The *left regular representation* L_A of $\mathbf{M}(G)$ is a representation of $\mathbf{M}(G)$ by bounded operators on the Hilbert space $L_2(G)$ defined by $L_A(\mu)f = \mu * f$.

The reason for our notation is that the representation L_A of $\mathbf{M}(G)$ is closely related to the representation L of G. In fact, a quick calculation shows that $L_A(\delta_a)f = L(a)f$ for all $a \in G$. It is the restriction of this representation to the subalgebra $\mathbf{M}_a(G)$ of $\mathbf{M}(G)$ that will be used to prove the existence of a large collection of continuous unitary representations of G. The next theorem takes up the task of showing that the regular representation is in fact a well-defined faithful representation of G.

Theorem 68. The a faithful representation L_A of $\mathbf{M}(G)$ is a well-defined faithful $*$-representation of $\mathbf{M}(G)$ by bounded operators on $L_2(G)$.

Proof. For any $f \in L_2(G)$, we have that $\|L_A(\mu)f\| = \|\mu * f\| \leqslant \|\mu\|\|f\|_2$. It follows that $\|L_A(\mu)\| \leqslant \|\mu\|$, and so $L_A(\mu)$ is a bounded operator on $L_2(G)$. Moreover, the linearity of L_A follows immediately from the properties of convolution. If $f \in C_{00}(G)$, then

$$L_A(\mu * \nu)f = (\mu * \nu) * f = \mu * (\nu * f) = L_A(\mu)(\nu * f) = L_A(\mu)L_A(\nu)f.$$

Since $C_{00}(G)$ is dense in $L_2(G)$, it follows that $L_A(\mu * \nu)f = L_A(\mu)L_A(\nu)f$ for all $f \in L_2(G)$. Thus L_A is a representation of $\mathbf{M}(G)$ by bounded operators.

If $f, g \in C_{00}(G)$, then by Fubini's theorem, we have that

$$\begin{aligned}\langle \pi_A(\mu^*)f, h\rangle &= \int_G (\mu^* * f)(x)\overline{h(x)}dx = \int_G \overline{h(x)}\overline{\int_G \overline{f(yx)}d\mu(y)}dx \\ &= \overline{\int_G\int_G h(x)\overline{f(yx)}dxd\mu(y)} = \overline{\int_G\int_G h(y^{-1}x)\overline{f(x)}dxd\mu(y)} \\ &= \overline{\int_G \overline{f(x)}\int_G h(y^{-1}x)d\mu(y)dx} = \overline{\int_G \overline{f(x)}(\mu * h)(x)dx} = \langle f, \pi_A(\mu)h\rangle.\end{aligned}$$

Since $C_{00}(G)$ is dense in $L_2(G)$, this shows that $\langle \pi_A(\mu^*)f, h\rangle = \langle f, \pi_A(\mu)h\rangle$ for all $f, h \in L_2(G)$. Thus $(\pi_A(\mu^*))^* = \pi_A(\mu)$, and we conclude that L_A is a $*$-representation of $\mathbf{M}(G)$. Lastly, to show that the representation is faithful, let $0 \neq \mu \in \mathbf{M}(G)$, let θ be the continuous function $\theta(x) = x^{-1}$ and let $f \in C_{00}(G)$ be such that $\int_G f \circ \theta d\mu \neq 0$. Then $f \circ \theta$ is right uniformly continuous and

$$\begin{aligned}|\mu * f(y) - \mu * f(x)| &= \left|\int_G f(z^{-1}y) - f(z^{-1}x)d\mu(z)\right| \\ = \left|\int_G L(y)f\circ\theta - L(z^{-1})f\circ\theta d\mu(z)\right| &\leqslant \|L(y)f\circ\theta - L(z)f\circ\theta\|_u\|\mu\|,\end{aligned}$$

so that $\mu * f$ must also be continuous. Since in particular $(\mu * f)(e) = \int_G f \circ \theta d\mu \neq 0$, this allows us to conclude that $\|\mu * f\|_2 \neq 0$ and therefore $L_A(\mu)$ is not identically zero on $L_2(G)$. This proves our theorem. □

We are now fully prepared to prove the Gel'fand-Raĭkov theorem, which establishes the fact that the collection of all irreducible, unitary, continuous representations in some sense "separates points", i.e. that for every $x, y \in G$ with $x \neq y$ there are irreducible, unitary, continuous representations π and π' with $\pi(x) \neq \pi'(y)$. The result is fundamental and important in the sense that it may be combined later with a version of the Stone-Weierstrass theorem to obtain that the closed linear space of all functions of the form $x \mapsto \langle \pi(x)f, h\rangle$ are closed in $C_0(G)$ (and therefore $L_p(G)$).

Theorem 69 (Gel'fand-Raĭkov)**.** A locally compact group G has *sufficiently many* irreducible, unitary continuous representations. That is, for every non-identity element $g \in G$, there exists a continuous, irreducible unitary representation π of G with $\pi(g) \neq I$.

Proof. Let U be a symmetric neighborhood of e with $\lambda(U) < \infty$ and $g \notin U^2$. Then $aU \cap U = \varnothing$. Let μ be the measure in $\mathbf{M}_a(G)$ such that $d\mu = 1_U d\lambda$. Then $d(\delta_g * \mu) = L(g)1_U dx = 1_{gU} dx$ and $\|\delta_g * \mu - \mu\| = \|1_{gU} - 1_U\| = 2\lambda(U)$, and therefore $\delta_a * \mu \neq \mu$. Since L_A is faithful, it follows that $L_A(\delta_g * \mu) \neq \pi_A(\mu)$. For every $g \in G$, there exists an irreducible $*$-representation π_A of $M_a(G)$ over a Hilbert space H such that $\pi_A(\delta_g * \mu) \neq \pi_A(\mu)$ (see [2] pp. 330). By Theorem (66), there exists a continuous unitary representation π of G such that

$$\langle \pi_A(\nu)\psi, \phi \rangle = \int_G \langle \pi(x)\psi, \phi \rangle d\nu(x)$$

for all $\nu \in \mathbf{M}_a(G)$ and all $\psi, \phi \in H$. Furthermore,

$$\begin{aligned} \int_G \langle \pi(x)\psi, \phi \rangle d\delta_g * \mu(x) &= \int_G \int_G \langle \pi(yx)\psi, \phi \rangle d\delta_g(y) d\mu(x) \\ &= \int_G \langle \pi(gx)\psi, \phi \rangle d\mu(x) = \int_G \langle \pi(g)\pi(x)\psi, \phi \rangle d\mu(x) \end{aligned}$$

from which it follows that

$$\int_G \langle \pi(g)\pi(x)\psi, \phi \rangle d\mu(x) \neq \int_G \langle \pi(x)\psi, \phi \rangle d\mu(x).$$

If π is the identity, this is a contradiction, so $\pi \neq I$. Any subspace $V \subset H$ invariant with respect to the representation π is also invariant with respect to π_A by Theorem (60), and therefore 0 or H, since π_A is irreducible. It follows that π is irreducible. This proves our theorem. □

Of special importance to us are compact groups, which satisfy the property that all irreducible, unitary, continuous representations must be finite dimensional. The Gel'fand-Raĭkov theorem then tells us that there are sufficiently many representations of this kind. Similarly, for locally compact abelian groups, all the irreducible, unitary representations must be 1-dimensional, and the Gel'fand-Raĭkov theorem tells us we have sufficiently many of these as well.

Theorem 70. Every irreducible continuous representation π of a compact group G by unitary operators on a Hilbert space H is finite-dimensional.

Proof. Recall that for compact groups we use normalized Haar measure ($\lambda(G) = 1$). Let G be a compact group and π be an irreducible continuous representation of G by unitary operators on a Hilbert space H. For any $\psi, \varphi \in H$, the function $\langle \pi(\cdot)\varphi, \psi \rangle \in L_2(G)$. In fact,

$$\begin{aligned} |\langle \langle \pi(\cdot)\varphi, \psi\rangle, \langle \pi(\cdot)\varphi, \phi\rangle\rangle| &= \left| \int_G \langle \pi(x)\varphi, \psi\rangle \overline{\langle \pi(x)\varphi, \phi\rangle} \right| dx \\ &\leqslant \int_G \|\pi(x)\varphi\|^2 \|\psi\| \|\phi\| dx = \|\varphi\|^2 \|\psi\| \|\phi\|. \end{aligned}$$

By the properties of the inner product, $\langle \langle \pi(\cdot)\varphi, \psi\rangle, \langle \pi(\cdot)\varphi, \phi\rangle\rangle$ therefore a bounded functional that is linear in ϕ and conjugate-linear in ψ for fixed φ, and we can conclude that there exists a bounded operator B_φ on H such that

$$\langle B_\varphi \phi, \psi\rangle = \langle \langle \pi(\cdot)\varphi, \psi\rangle, \langle \pi(\cdot)\varphi, \phi\rangle\rangle.$$

for all $\phi, \psi \in H$. Furthermore, due to the invariance of the Haar integral with respect to left translation,

$$\begin{aligned}
\langle B_\varphi \pi(a)\phi, \psi\rangle &= \langle \langle \pi(\cdot)\varphi, \psi\rangle, \langle \pi(\cdot)\varphi, \pi(a)\phi\rangle\rangle = \int_G \langle \pi(x)\varphi, \psi\rangle \overline{\langle \pi(x)\varphi, \pi(a)\phi\rangle} dx \\
&= \int_G \langle \pi(a^{-1})\pi(x)\varphi, \pi(a^{-1})\psi\rangle \overline{\langle \pi(a^{-1})\pi(x)\varphi, \phi\rangle} dx \\
&= \int_G \langle \pi(a^{-1}x)\varphi, \pi(a^{-1})\psi\rangle \overline{\langle \pi(a^{-1}x)\varphi, \phi\rangle} dx \\
&= \int_G \langle \pi(x)\varphi, \pi(a^{-1})\psi\rangle \overline{\langle \pi(x)\varphi, \phi\rangle} dx \\
&= \langle \langle \pi(\cdot)\varphi, \pi(a^{-1})\psi\rangle, \langle \pi(\cdot)\varphi, \phi\rangle\rangle = \langle B_\varphi \phi, \pi(a^{-1})\psi\rangle = \langle \pi(a) B_\varphi \phi, \psi\rangle.
\end{aligned}$$

Since $\phi, \psi \in H$ were taken arbitrarily, this shows that B_φ commutes with $\pi(a)$ for all $a \in G$. Since π is irreducible, it follows that $B_\varphi = \alpha_\varphi I$ for some $\alpha_\varphi \in \mathbb{C}$. In particular, this shows us that

$$\alpha_\varphi \|\psi\|^2 = \langle B_\varphi \psi, \psi\rangle = \|\langle \pi(\cdot)\varphi, \psi\rangle\|_2^2$$

for all $\psi, \varphi \in H$. Let θ be the continuous function on G defined by $\theta(x) = x^{-1}$. Reversing φ and ψ above, then gives us

$$\begin{aligned}
\alpha_\psi \|\varphi\|^2 &= \|\langle \pi(\cdot)\psi, \varphi\rangle\|_2^2 = \|\langle \varphi, \pi(\cdot)\psi\rangle\|_2^2 = \|\langle \pi(\cdot)\varphi, \psi\rangle \circ \theta\|_2^2 \\
&= \|\langle \pi(\cdot)\varphi, \psi\rangle\|_2^2 = \alpha(\varphi)\|\psi\|^2,
\end{aligned}$$

where we have used the fact that the Haar integral is invariant under inversion for unimodular groups (in particular, for compact groups). Now if we take $\|\varphi\| = 1$, and set $c = \alpha_\varphi$ we find that $\alpha_\psi = c\|\psi\|^2$ for all $\psi \in H$. The function $|\langle \pi(\cdot)\varphi, \varphi\rangle|^2$ is continuous and is equal to $\|\varphi\|^2 = 1$ at e. The above then also tells us that $0 < \|\langle \pi(\cdot)\varphi, \varphi\rangle\|_2^2 = \alpha_\varphi \|\varphi\|^2 = c$, so that necessarily $c > 0$.

Now if we take any collection of orthonormal vectors $\{\xi_i\}_{i=1}^n \subset H$, we have that

$$\|\langle\pi(\cdot)\xi_i, \xi_j\rangle\|_2^2 = \alpha_{\xi_i}\|\xi_j\|^2 = c\|\xi_i\|^2\|\xi_j\|^2 = c.$$

Given $x \in G$, since $\pi(x)$ is unitary, the collection $\{\pi(x)\xi_i\}_{i=1}^n$ is orthonormal and may be extended to an orthonormal basis B of H. We then have the inequality

$$\sum_{i=1}^n |\langle\pi(x)\xi_i, \psi\rangle|^2 \leqslant \sum_{\phi\in B} |\langle\phi, \psi\rangle|^2 = \|\phi\|^2,$$

which holds for any $x \in G$ and $\phi \in H$. In particular, taking $\phi = \xi_j$, we find

$$nc = \sum_{i=1}^n \|\langle\pi(\cdot)\xi_i, \xi_j\rangle\|_2^2 \leqslant \int_G \sum_{i=1}^n |\langle\pi(\cdot)\xi_i, \xi_j\rangle|^2 \leqslant \int_G \|\xi_j\|^2 = 1.$$

In particular, this says that $n \leqslant 1/c$, so that n must be finite. This proves our theorem. □

Corollary 71. Let G be a compact group. Then G has sufficiently many irreducible, unitary continuous representations by unitary matrices.

Proof. This follows immediately from the Gel'fand-Raĭkov theorem and the fact that all irreducible continuous unitary representations of G are finite dimensional. □

Definition 72. Let S be a semigroup. A homomorphism of S into $\mathbb{C}$ is called a *multiplicative function on* S. A nontrivial bounded multiplicative function on S is called a *semicharacter of* S. A semicharacter of a group G is called a *character* of G.

The assumption that a semicharacter of a semigroup S is bounded requires that it maps S into the disk $\{\lambda \in \mathbb{C} : |\lambda| \leqslant 1\}$. Moreover if χ is a character of a group G, then χ is a homomorphism of G into $\mathbb{T}$, since if for some $g \in G$ $|\chi(g)| < 1$, then $1 = |\chi(e)| = |\chi(g^{-1})\chi(g)| = |\chi(g^{-1})||\chi(g)|$, so that $\chi(g^{-1}) > 1$, which is a contradiction. Thus every character of G is a 1-dimensional unitary representation of G over the Hilbert space $\mathbb{C}$ and every 1-dimensional unitary representation π of G by operators on a Hilbert space defined by the orthonormal basis $\{f\}$ defines a character χ via the relation $\pi(g)(f) = \chi(g)f$. This allows us to identify the collection of all characters of G with the collection of all 1-dimensional unitary representations of G.

Theorem 73. Every irreducible continuous unitary representation π of a locally compact abelian group G is 1-dimensional.

Proof. Since G is abelian, for any $a \in G$, $\pi(a)$ commutes with $\pi(x)$ for all $x \in G$. Since π is irreducible, it follows that $\pi(a) = \lambda I$ for some $\lambda \in \mathbb{C}$. Every subspace of H is invariant under an operator of the form λI, so H must be 1-dimensional. □

Corollary 74. Every locally compact abelian group G has sufficiently many continuous characters.

Proof. This is an immediate consequence of the Gel'fand-Raĭkov theorem and the fact that all irreducible, continuous, unitary representations of G are 1-dimensional. □

CHAPTER 3. UNITARY REPRESENTATIONS OF COMPACT GROUPS

3.1. Introduction

The purpose of this chapter is to set up the machinery necessary to do harmonic analysis on compact groups, and to prove the famous Peter-Weyl theorem that establishes that the coordinate functions of a compact group G determine a dense orthonormal set. For the most part, the notation from the previous chapter will carry over. However, in this chapter we will always specifically assume that G is compact unless otherwise stated. It follows from Theorem (69) that the irreducible, continuous, unitary operators on G will always be finite-dimensional.

3.2. Basic Definitions and Facts

We begin this chapter as well with a rather large collection of definitions to set the stage.

Definition 75. Let S be a semigroup and X and X' reflexive Banach spaces. Let π and ρ be two representations of S by bounded operators on X and X' respectively. A bounded linear transformation $T : X \to X'$ satisfying $T\pi(g) = \rho(g)T$ for all $g \in S$ is called an *intertwining operation.* The collection of all intertwining operations on G is a closed linear subspace of the Banach space $B(X, X')$ of all bounded linear transformations from X into X' and is denoted by $\text{Twist}(\pi, \rho)$.

It is clear that two representations are equivalent if $\text{Twist}(\pi, \rho)$ contains a linear isometry. Moreover, if $\pi = \rho$, then $\text{Twist}(\pi, \rho) = \{\alpha I : \alpha \in \mathbb{C}\}$. (see Corollary (92)).

Definition 76. The equivalence of irreducible unitary operators defined above is an equivalence relation, and we call the collection of equivalence classes under this relation the *dual object* of G and denote it by Σ. For any given equivalence class $\sigma \in \Sigma$, we write π_σ for a representation in σ and H_σ for its corresponding Hilbert space.

The equivalence class of a continuous character of G will be a singleton set, and so we may identify the subset of Σ of all 1-dimensional representations with the collection $\mathfrak{X}$ of all continuous characters on G.

Definition 77. For a compact group G and a fixed $\sigma \in \Sigma$, it is obvious that all $\pi_\sigma \in \sigma$ operate on Hilbert spaces of the same finite dimension. We let d_σ denote the dimension of these representation spaces. We denote by 1 both the trivial 1-dimensional representation of G and its corresponding equivalence class. Like all 1-dimensional representations, its equivalence class consists of itself alone. Given an element $\sigma \in \Sigma$ with representation $\pi_\sigma \in \sigma$ and a basis $\{\xi_i\}_{i=1}^{d_\sigma}$ for H_σ, we define a collection of functions $\pi_{jk}^{(\sigma)} : G \to \mathbb{C}$ by

$$\pi_{jk}^{(\sigma)}(g) = \langle \pi_\sigma(g)\xi_k, \xi_j \rangle$$

for all $g \in G$. The functions $\pi_{jk}^{(\sigma)}$ are called *coordinate functions* for $\pi_\sigma \in \sigma$ and the basis $\{\xi_1, \ldots, \xi_{d_\sigma}\}$. In particular, $\pi_{jk}^{(\sigma)}(g)$ is the matrix of the operator $\pi_\sigma(g)$ in this basis. We note that the mapping $g \mapsto \pi_{jk}^{(\sigma)}(g)$ is a representation of g by unitary matrices. In particular, we have the relations

(i)

$$\pi_\sigma(g)\xi_k = \sum_{j=1}^{d_\sigma} \pi_{jk}^{(\sigma)}(g)\xi_j$$

(ii)

$$\pi_{jk}^{(\sigma)}(gh) = \sum_{l=1}^{d_\sigma} \pi_{jl}^{(\sigma)}(g)\pi_{lk}^{(\sigma)}(h)$$

(iii)

$$\pi_{jk}^{(\sigma)}(g^{-1}) = \overline{\pi_{kj}^{(\sigma)}(g)}$$

Definition 78. For $\sigma \in \mathfrak{P}$ and $\pi_\sigma \in \sigma$ with representation space H_σ, we define $\mathrm{Trig}_\sigma(G)$ to be the collection of all finite complex linear combinations of functions of the form

$$\langle \pi_\sigma(\cdot)\psi, \phi\rangle$$

with $\psi, \phi \in H_\sigma$.

The definition of $\mathrm{Trig}_\sigma(G)$ is independent of the choice of representation π_σ. In fact, if $\rho \in \sigma$ is another representation with representation space H'_σ, there exists a linear isometry $T : H_\sigma \to H'_\sigma$ such that $\rho(g)T = T\pi_\sigma(g)$ for all $g \in G$ and therefore any function of the above form can be rewritten as

$$g \mapsto \langle \pi_\sigma(g)\psi, \phi\rangle = \langle T\pi_\sigma(g)\psi, T\phi\rangle = \langle \rho(g)T\psi, T\phi\rangle = \langle \rho(g)\psi', \phi'\rangle$$

where $\psi' = T\psi$ and $\phi' = T\phi$ are vectors in H'_σ. Conversely, in a similar way any function of the form $g \mapsto \langle \rho(g)\psi', \phi'\rangle$ for $\psi', \phi' \in H'_\sigma$ is expressable in the form $g \mapsto \langle \pi_\sigma(g)\psi, \phi\rangle$ for some $\psi, \phi \in H_\sigma$.

If $\mathfrak{p} \subset \mathfrak{P}$, then we denote by $\mathrm{Trig}_{\mathfrak{p}}(G)$ the smallest linear space of functions containing $\bigcup_{\sigma\in\mathfrak{p}} \mathrm{Trig}_\sigma(G)$. If $\mathfrak{p} = \mathfrak{P}$, we write $\mathrm{Trig}_{\mathfrak{p}}(G)$ as $\mathrm{Trig}(G)$. Functions in $\mathrm{Trig}(G)$ are called *trigonometric polynomials* on G.

The justification for calling functions $\mathrm{Trig}(G)$ trigonometric polynomials, stems from the special case of $G = T$. In this case, $\mathfrak{P} \equiv \mathfrak{X} = \{g \mapsto g^k : k \in \mathbb{Z}\}$. Thus the functions $g \mapsto \langle \pi(g)\xi, \eta\rangle$ all have the form $g \mapsto \alpha g^n$, for some $\alpha \in \mathbb{C}$ and $n \in \mathbb{Z}$. Since each g is of the form $g = e^{it}$, $\mathrm{Trig}(T)$ consists of all functions of the form $\sum_{k=-n}^{n} \alpha_k \exp(ikt)$, which is the usual collection of trigonometric polynomials.

We now begin to outline some of the properties of trigonometric polynomials.

Fact 79. The function spaces $\mathrm{Trig}_\sigma(G)$ and $\mathrm{Trig}_{\mathfrak{p}}(G)$ for $\sigma \in \mathfrak{P}$ and $\mathfrak{p} \subset \mathfrak{P}$ are invariant under left and right translation. In fact, for any representation π of G by unitary operators on a Hilbert space H

$$(R(a)\langle\pi(\cdot)\psi,\phi\rangle)(g) = \langle\pi(ga)\psi,\phi\rangle = \langle\pi(g)(\pi(a)\psi),\phi\rangle;$$
$$(L(a)\langle\pi(\cdot)\psi,\phi\rangle)(g) = \langle\pi(ag)\psi,\phi\rangle = \langle\pi(g)\psi,\pi(a^{-1})\phi\rangle.$$

3.3. Trigonometric Polynomials and Stone-Weierstrass

Since representations $\pi_\sigma \in \sigma \in \mathfrak{p}$ are continuous, they are weakly continuous and therefore $\mathrm{Trig}_{\mathfrak{p}}(G) \subset C(G) = C_{00}(G)$. Moreover, as long as $\mathfrak{p}$ is closed under certain operations $\mathrm{Trig}_{\mathfrak{p}}(G)$ will actually be an algebra of functions. This is a very helpful property, since it will allow us to apply the Stone-Weierstrass theorem and obtain the property that $\mathrm{Trig}_{\mathfrak{p}}(G)$ is dense in $C_{00}(G)$. In this section, we establish the requirements necessary for $\mathrm{Trig}_{\mathfrak{p}}(G)$ to be an algebra, and in particular show that $\mathrm{Trig}(G)$ is dense in $C_{00}(G)$. We begin with some definitions.

Definition 80. Let H be a Hilbert space of dimension d, and let $\{\xi_i\}_{i=1}^d$ be an orthonormal basis of H. Define a mapping $D : H \to H$ by

$$D\left(\sum_{i=1}^{d}\langle\psi,\xi_i\rangle\xi_i\right) = \sum_{i=1}^{d}\overline{\langle\psi,\xi_i\rangle}\xi_i. \tag{23}$$

A transformation of the form D above is called a *conjugation transformation.*

The properties of conjugate transformations are outlined in the next lemma.

Lemma 81. Let D be a conjugate transformation of a d-dimensional Hilbert space H. Then D is a continuous conjugate-linear bijection on H with $D^2 = I$. Let $T : H \to H'$ be a linear operator from H to another Hilbert space H'. Then DTD is also a linear operator. If T is unitary, then DTD is unitary. If T is continuous, so too is DTD. If $H = H'$ and V is a T-invariant subspace of H, then DV is DTD-invariant. If T is a linear isometry, then DT is a conjugate transformation of H. If D' is another conjugate transformation of H, then there exists a linear isometry L on H such that $D' = DL$.

Proof. Let D be a conjugate transformation of H and let $\{\xi_i\}_{i=1}^d$ be an orthonormal basis for H satisfying Equation (23). The fact that D is conjugate-linear and bijective follows immediately from its definition. The fact that DTD is linear and $D^2 = I$ follows immediately from the definition and an obvious calculation, since elements are conjugated twice. Moreover

$$\|D\psi\|^2 = \left\| \sum_{i=1}^{d} \overline{\langle \psi, \xi_i \rangle} \xi_i \right\|^2 \leqslant \sum_{i=1}^{d} |\overline{\langle \psi, \xi_i \rangle}|^2 = \|\psi\|,$$

so D is norm-preserving, and therefore continuous. In particular, since D is norm-preserving, DTD is unitary if T is. The composition of continuous mappings is continuous, so DTD is continuous when T is. If $H = H'$ and V is a subspace of H invariant under T, then any $D\psi \in DV$ satisfies $DTD(D\psi) = DT\psi \in DV$. It follows that DV is a DTD-invariant subspace of H. The last two properties of the theorem above follow simply from the properties of change of base in a Hilbert space. □

From the above lemma, we immediately obtain the following theorem.

Theorem 82. Let π be a continuous, irreducible, unitary representation of G with representation space H, and let D be a conjugate transformation on H. Then $D\pi D : x \mapsto D\pi(x)D$ is also a continuous, irreducible, unitary representation of G. If D' is another conjugate transformation of H and ρ is equivalent to π, then $D'\rho D'$ is equivalent to $D\pi D$. In particular, $D'\pi D'$ is equivalent to $D\pi D$.

Proof. From the previous lemma, we know that $D = D^{-1}$. If $x, y \in G$, then $D\pi(xy)D = D\pi(x)\pi(y)D = D\pi(x)DD\pi(y)D$, and so $D\pi D$ is a representation of G. From the above, we also know that $D\pi D$ is continuous. The lemma above also shows that for any $x \in G$, any $D\pi(x)D$-invariant subspace of $V \subset H$ corresponds to a $\pi(x)$ invariant subspace DV, and since D is a bijection, this allows us to conclude that $D\pi D$ is irreducible. The previous lemma also shows that $D\pi D$ is unitary if π is. If D' is another conjugate transformation of H and ρ is equivalent to π with corresponding representation space H', then there exists isometries $T : H \to H'$ and $L : H \to H$ such that $T\pi(g) = \rho(g)T$ for all $g \in G$ and $DL = D'$. Since $D = D^{-1}$ and $D' = (D')^{-1}$, we also have that $L^{-1}D = D'$. It follows that

$$D'\rho(g)D' = DL\rho(g)L^{-1}D = DLT\pi(g)T^{-1}L^{-1}D = DLTD(D\pi(g)D)(DLTD)^{-1}.$$

By the previous lemma, $DLTD$ is an isometry from H to H'. This proves our theorem. □

The theorem we just proved allows us to justify the following definition, which defines a conjugation operation on $\mathfrak{P}$.

Definition 83. Let π be a representation of G by unitary operators on a Hilbert space H. A representation of the form $D\pi D : x \mapsto D\pi(x)D$ called a *conjugate representation* of π. Let $\sigma \in \mathfrak{P}$. The *conjugate of the dual element* $\overline{\sigma}$ is the member of $\mathfrak{P}$ given by the equivalence class of all representations conjugate to a representation $\pi_\sigma \in \sigma$. A subset $\mathfrak{p} \subset \mathfrak{P}$ is *closed under conjugation* if $\overline{\sigma} \in \mathfrak{p}$ for all $\sigma \in \mathfrak{p}$.

The importance of the conjugates of dual elements, follows from the fact that if $\mathfrak{p} \subset \mathfrak{P}$ is closed under the conjugation operation defined above, then $\mathrm{Trig}_{\mathfrak{p}}(G)$ is closed under conjugation as well. To show this, take any $\pi_\sigma \in \sigma \in \mathfrak{P}$ with corresponding representation space H_σ. Then fix a basis $\{\xi_i\}_{i=1}^{d_\sigma}$ of H_σ, let $\pi_{ij}^{(\sigma)}$ denote the usual coordinate functions with respect to this basis, and let D be the conjugation operation on H with respect to this basis. Then $D\xi_i = \xi_i$ for all i and

$$\langle D\pi(g)D\xi_i, \xi_j\rangle = \overline{\langle DD\pi(g)D\xi_i, D\xi_j\rangle} = \overline{\langle \pi(g)\xi_i, \xi_j\rangle} = \overline{\pi_{ij}^{(\sigma)}}.$$

Thus the coordinate functions of $D\pi D$ with respect to the basis $\{\xi_i\}_{i=1}^{d_\sigma}$ are the complex conjugates of those of π.

Now to establish $\mathrm{Trig}_{\mathfrak{p}}(G)$ as an algebra, it remains only to show that it is closed under products. This requires that we have $\mathfrak{p}$ closed under an additional operation, namely tensor product. We have the following definition.

Definition 84. Let π_1 and π_2 be two irreducible, continuous, unitary representations G with corresponding Hilbert spaces H_1 and H_2, respectively. The *tensor product of the representations* $\pi_1 \otimes \pi_2$ is the representation of G defined by $\pi_1 \otimes \pi_2(g) = \pi_1(g) \otimes \pi_2(g)$ that acts on the Hilbert space $H_1 \otimes H_2$[10].

The tensor product has many useful properties, which we do not prove here. The properties that we will need are that the tensor product of two continuous or unitary operators has the same property. In particular, the tensor product of two continuous, unitary representations is continuous and unitary. It is not true, however, that the tensor product of two irreducible representations is irreducible. Instead, we must consider the decomposition of the Hilbert space $H_1 \otimes H_2$ into $\pi_1 \times \pi_2$-invariant subspaces. If H_1 has dimension d_1 and H_2 has dimension d_2, then the dimension of $H_1 \otimes H_2$ is $d_1 d_2$, and therefore finite. Thus there are no problems with simply taking a decomposition

$$H_1 \otimes H_2 = \sum_{i=1}^{n} m_i H_i', \tag{24}$$

where H_i' is a $\pi_1 \times \pi_2$-invariant subspace of $H_1 \otimes H_2$ for all i and we use the notation mH to denote the direct sum $H \oplus H \oplus \ldots H$ of m copies of a Hilbert space H. Defining

[10] The tensor product of two Hilbert spaces is always over the field $\mathbb{C}$.

π_i' to be the restriction of $\pi_1 \otimes \pi_2$ to H_i', this decomposes the representation π into the direct sum of representations

$$\pi_1 \otimes \pi_2 = \sum_{i=1}^{n} m_i \pi_i', \tag{25}$$

where we have used the notation $m\pi$ to denote the direct sum $\pi \oplus \pi \oplus \ldots \pi$ of m copies of the representation π. If the representations π_1 and π_2 (and therefore $\pi_1 \otimes \pi_2$) are continuous and unitary, then the restrictions π_i' surely are, too. Moreover n is finite and m_i is finite for each i, since the dimension of $H_1 \otimes H_2$ is finite. We have the following definition.

Definition 85. A subset $\mathfrak{p} \subset \mathfrak{P}$ is *closed under tensor product* if given any $\sigma_i \in \mathfrak{P}$ and $\pi_i \in \sigma_i$ with corresponding representation space H_i for $i = 1, 2$, the equivalence classes of the elements π_i' of the decomposition given in Equation (24) and Equation (25) are contained in $\mathfrak{p}$.

In particular $\mathfrak{P}$ is closed under tensor products by definition. If $\mathfrak{p}$ is closed under tensor product, then take any $\sigma_i \in \mathfrak{P}$ and $\pi_i \in \sigma_i$ with corresponding representation space H_i for $i = 1, 2$. For any $\psi_i, \phi_i \in H_i$, we have that

$$\langle (\pi_1 \otimes \pi_2)(\psi_1 \otimes \psi_2), \phi_1 \otimes \phi_2 \rangle = \langle \pi_1\psi_1 \otimes \pi\psi_2, \phi_1 \otimes \phi_2 \rangle = \langle \pi_1\psi_1, \phi_1 \rangle \langle \pi_2\psi_2, \phi_2 \rangle$$

by definition. Closure under tensor products means that $\langle (\pi_1 \otimes \pi_2)(\psi_1 \otimes \psi_2), \phi_1 \otimes \phi_2 \rangle$ can be written as a linear combination of members of $\mathrm{Trig}_{\mathfrak{p}}(G)$, and therefore $\mathrm{Trig}_{\mathfrak{p}}(G)$ is closed under products. It follows that if $\mathfrak{p}$ is closed under tensor products and conjugation that $\mathrm{Trig}_{\mathfrak{p}}(G)$ is an algebra.

Recall the Stone-Weierstrass theorem.

Theorem 86 (Stone-Weierstrass)**.** Let S be a compact Hausdorff space and $C(S)$ be the algebra of all real continuous functions on S. Let A be a closed subalgebra of

$C(S)$ containing the identity. Then $A = C(S)$ if and only if A separates the points of S.

A proof of this theorem will not be given here, but can be found many places. See, for example, [6] pp. 272. We use the following definition.

Definition 87. A subset $\mathfrak{p} \subset \mathfrak{P}$ *separates points* in G if for all $x \in G$ with $x \neq e$ there exists $\sigma \in \mathfrak{p}$ such that $\pi_\sigma(x) \neq I$ for some $\pi_\sigma \in \sigma$.

This is equivalent to the usual definition of "separates points", in that $x \neq y$ if and only if $xy^{-1} \neq e$, which is true if and only if $\pi_\sigma(xy^{-1}) \neq I$ for some $\pi_\sigma \in \sigma \in \mathfrak{p}$.

The consequence of this theorem is immediate: since the Gel'fand-Raĭkov theorem tells us that members of equivalence classes of $\mathfrak{P}$ distinguish points in the compact Hausdorff space G, the Stone-Weierstrass theorem tells us the closure of $\mathrm{Trig}_{\mathfrak{p}}(G)$ is $C(G)$. Specifically, we have the following theorem.

Theorem 88. The trigonometric polynomials of G are dense in $C(G)$.

Proof. We have that $\mathfrak{P}$ separates points by the Gel'fand-Raĭkov theorem. Our theorem then follows from the Stone-Weierstrass theorem and the fact that $\mathrm{Trig}(G)$ is an algebra. □

An interesting question might be whether or not there is a smaller subcollection $\mathfrak{p}$ of $\mathfrak{P}$ will be closed under tensor product and conjugation and also separates points in G so that $\mathrm{Trig}_{\mathfrak{p}}(G)$ is dense in $C(G)$. The answer turns out to be no, though we do not yet have the tools to prove it. These are developed by the orthogonality relations, taken up in the next section.

3.4. Orthogonality Relations

In this section we establish the orthogonality relations between the coordinate functions.

Theorem 89. Let π_1 and π_1 be unitary representations of an arbitrary group G with representation spaces H_1 and H_2 respectively. Suppose that π_1 is irreducible and that there is a bounded linear isomorphism $T \in \text{Twist}(\pi_1, \pi_2)$. Then there exists a positive real number β such that βT is a linear isometry from H_1 to H_2 and $\pi_1 \sim \pi_2$.

Proof. Since $T \in \text{Twist}(\pi_1, \pi_2)$, for every $g \in G$ the following diagram commutes.

$$\begin{array}{ccc} H_1 & \xrightarrow{\pi_1(g)} & H_1 \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle T} \\ H_2 & \xrightarrow{\pi_2(g)} & H_2 \end{array} \quad \text{or equivalently} \quad \begin{array}{ccc} H_2 & \xrightarrow{\pi_2(g)} & H_2 \\ {\scriptstyle T^{-1}}\downarrow & & \downarrow{\scriptstyle T^{-1}} \\ H_1 & \xrightarrow{\pi_1(g)} & H_1. \end{array}$$

If W is a π_2-invariant subspace, then $\pi_2(g)(W) \subset W$ for all $g \in G$ and therefore $\pi_1(g)(T^{-1}(W)) = T^{-1}(\pi_2(g)(W)) \subset T^{-1}(W)$. Thus $T^{-1}(W)$ is π_1-invariant and must be 0 or H_1, since π_1 is irreducible. Therefore W must be 0 or H_2. We conclude that π_2 is irreducible.

Define $T^* : H_2 \to H_1$ by

$$\langle T^*\psi_2, \psi_1 \rangle_1 = \langle \psi_2, T\psi_1 \rangle_2,$$

for all $\psi_1 \in H_1, \psi_2 \in H_2$. This mapping is well-defined: For fixed ψ_2, the right side of the above equation is a conjugate-linear ($\lambda x \mapsto \overline{\lambda} T^* x$, $\lambda \in \mathbb{C}$) and continuous functional on H_1 and therefore has the form $\langle \phi, \psi_1 \rangle_1$ for some unique $\phi \in H_1$. We take $\phi = T^*\psi_2$. It is routine to show the mapping T^* is a bounded linear isomorphism from H_2 to H_1.

For $g \in G$, we also notice that for any $\psi_1 \in H_1$ and $\psi_2 \in H_2$,

$$\begin{aligned} \langle \pi_1(g)T^*\psi_2, \psi_1 \rangle_1 &= \langle T^*\psi_2, \pi_1(g^{-1})\psi_1 \rangle_1 = \langle \psi_2, T\pi_1(g^{-1})\psi_1 \rangle_2 \\ &= \langle \psi_2, \pi_2(g^{-1})T\psi_1 \rangle_2 = \langle \pi_2(g)\psi_2, T\psi_1 \rangle_2 = \langle T^*\pi_2(g)\psi_2, \psi_1 \rangle_1, \end{aligned}$$

so that $\pi_1(g)T^* = T^*\pi_2(g)$. Since g was arbitrary, this tells us that the adjoint $T^* \in \text{Twist}(\pi_2, \pi_1)$. Now for any $g \in G$, $T^*T\pi_1(g) = T^*\pi_2(g)T = \pi_1(g)T^*T$, so that $T^*T \in \text{Twist}(\pi_1, \pi_1) = \{\alpha I : \alpha \in \mathbb{C}\}$. Thus $T^*T = \lambda I$ for some $\lambda \in \mathbb{C}$. Since $T^*T \neq 0$, $\lambda \neq 0$ and for any $\psi_1, \phi \in H_1$,

$$\lambda\langle \phi, \psi_1\rangle_1 = \langle T^*T\phi, \psi_1\rangle_1 = \langle T\phi, T\psi_1\rangle_2.$$

If $\phi = \psi_2 \neq 0$, the above tells us $\lambda = \|T\psi_1\|_2^2/\|\psi_1\|_1^2 > 0$. Thus taking $\beta = 1/\sqrt{\lambda}$ proves the theorem. $\square$

Theorem 90. Let π_1 and π_2 be continuous unitary representations of G with corresponding representation spaces H_1 and H_2 and let $B \in B(H_1, H_2)$. Then the mapping $A_B : H_1 \to H_2$ defined by

$$\langle A_B\psi, \phi\rangle_2 = C_B(\psi, \phi) = \int_G \langle \pi_2(g^{-1})B\pi_1(g)\psi, \phi\rangle_2 dg \tag{26}$$

(where the integral is taken with respect to the normalized Haar measure) is a bounded linear transformation from H_1 into H_2. Moreover $A_B \in \text{Twist}(\pi_1, \pi_2)$.

Proof. Let $\psi \in H_1$ and $\phi \in H_2$. We first show that $\langle \pi_2(g^{-1})B\pi_1(g)\psi, \phi\rangle_2$ is a continuous function on G. To show this, write

$$\begin{aligned}
&|\langle \pi_2(x^{-1})B\pi_1(x)\psi, \phi\rangle_2 - \langle \pi_2(y^{-1})B\pi_1(y)\psi, \phi\rangle_2|\\
\leqslant&|\langle B\pi_1(x)\psi, (\pi_2(x) - \pi_2(y))\phi\rangle_2| + |\langle B(\pi_1(x) - \pi_1(y))\psi, \pi_2(y)\phi\rangle_2|\\
\leqslant&\|B\| \cdot \|\psi\|_1 \cdot \|\pi_2(x)\phi - \pi_2(y)\phi\| + \|B\| \cdot \|\pi_1(x)\psi - \pi_1(y)\psi\| \cdot \|\phi\|_2\\
\leqslant&\|B\| \cdot \|\psi\|_1 \cdot \|\pi_2(x) - \pi_2(y)\| \cdot \|\phi\|_2 + \|B\| \cdot \|\pi_1(x) - \pi_1(y)\| \cdot \|\psi\|_1 \cdot \|\phi\|_2.
\end{aligned}$$

Since π_1 and π_2 are continuous, this shows $\langle \pi_2(g^{-1})B\pi_1(g)\psi, \phi\rangle_2$ to be continuous. It follows that the integral in Equation (26) is well-defined. Clearly C is a conjugate-linear functional on $H_1 \oplus H_2$. Moreover, for fixed $\psi \in H_1$ $C(\psi, \cdot)$ is a continuous conjugate-linear functional on H_2. Therefore there is a unique $\varphi \in H_2$ such that $\langle \varphi, v\rangle_2 = C(\psi, v)$ for all $\phi \in H_2$, and we define $A_B\psi = \varphi$. Clearly $A_B\psi$ is a linear transformation from H_1 into H_2. Furthermore, A_B is bounded, since

$$\begin{aligned}
\|A_B\psi\|_2^2 = \langle A_B\psi, A_B\psi\rangle_2 = C(\psi, A_B\psi) &= \int_G \langle \pi_2(g^{-1})B\pi_1(g)\psi, A_B\psi\rangle_2 dg \\
&\leqslant \sup_{g\in G}\{\|B\pi_1(g)\psi\|_2 \cdot \|A_B\psi\|_2\} \leqslant \|B\| \cdot \|\psi\|_1 \cdot \|A_B\psi\|_2.
\end{aligned}$$

In particular, this shows $\|A_B\| \leqslant \|B\|$. Now if we note that

$$\begin{aligned}
\langle A_B\pi_1(h)\psi, \phi\rangle_2 &= \int_G \langle \pi_2(g^{-1})B\pi_1(g)\pi_1(h)\psi, \phi\rangle_2 dg = \int_G \langle \pi_2(g^{-1})B\pi_1(gh)\psi, \phi\rangle_2 dg \\
&= \int_G \langle \pi_2((gh)^{-1})B\pi_1(gh)\psi, \pi_2(h^{-1})\phi\rangle_2 dg \\
&= \int_G \langle \pi_2(g^{-1})B\pi_1(g)\psi, \pi_2(h^{-1})\phi\rangle_2 dg \\
&= \langle A_B\psi, \pi_2(h^{-1})\phi\rangle_2 = \langle \pi_2(h)A_B\psi, \phi\rangle.
\end{aligned}$$

Since $\psi \in H_1$ and $\phi \in H_2$ are arbitrary, this shows that $A_B\pi_1(h) = \pi_2(h)A_B$ for all $h \in G$. Thus $A_B \in \text{Twist}(\pi_1, \pi_2)$. □

Lemma 91 (Schur's Lemma)**.** Let $\{S_\alpha\}_{\alpha\in\Omega}$ and $\{T_\alpha\}_{\alpha\in\Lambda}$ be irreducible sets of linear operators on vector spaces U and V respectively over a field k, and suppose that there is a linear transformation $f : U \to V$ such that for all $\alpha \in \Lambda$ there exists a $\beta \in \Omega$ such that the following diagram commutes

$$\begin{array}{ccc}
U & \xrightarrow{S_\beta} & U \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\
V & \xrightarrow{T_\alpha} & V,
\end{array}$$

and vise-versa. Then either $f = 0$ or f is bijective.

Proof. Consider $f(U)$. This is clearly a subspace of V and either $f(U)$ is 0 or V or there exists $\alpha \in \Lambda$ such that $T_\alpha(f(U)) \not\subseteq f(U)$. In the third case, there is a $\beta \in \Omega$ such that $f(S_\beta(U)) = T_a(f(U))$, and therefore $f(S_\beta(U)) \not\subseteq f(U)$. Since $S_\beta(U) \subset U$, this is a contradiction and we must have one of the first two cases. In the first case, $\operatorname{Ker} f = U$. In the second case, we note that

$$fS_\beta(\operatorname{Ker} f) = T_\alpha f(\operatorname{Ker} f) = T_\alpha(\{0\}) = 0$$

so that in particular $S_\beta(\operatorname{Ker} f) \subset \operatorname{Ker} f$ for all $\beta \in \Omega$. Since the collection $\{S_\alpha\}_{\alpha\in\Omega}$ is irreducible and $\operatorname{Ker} f \neq U$, we conclude that $\operatorname{Ker} f = 0$. Thus either $f = 0$ or f is a bijection. □

Corollary 92. Let $\{S_\alpha\}_{\alpha\in\Omega}$ be an irreducible set of linear operators on a vector spaces U over an algebraically closed field k, and suppose that there is a linear transformation $f : U \to U$ such that $fS_\alpha = S_\alpha f$ for all $\alpha \in \Omega$. Then $f = \lambda I$ for some $\lambda \in k$[11].

Proof. The function $x \mapsto \det(f - xI)$ is a polynomial in x over the field k. Since k is assumed to be algebraically closed, there is a root $\lambda \in k$. Clearly $(f - \lambda I)S_\alpha = S_\alpha(f - \lambda I)$ for all $\alpha \in \Omega$. By Schur's lemma, $f - \lambda I$ is either a bijection or 0. However, since $\det(f - \lambda I) = 0$, we know that it cannot be a bijection. We conclude $f = \lambda I$. □

Corollary 93. Let π be an irreducible continuous unitary representations of G with corresponding representation spaces U. Then $\operatorname{Twist}(\pi, \pi) = \{\alpha I\}_{\alpha\in\mathbb{C}}$.

Proof. Since $\mathbb{C}$ is algebraically closed, this follows immediately from Equation (92). □

Theorem 94. Let π and ρ be irreducible continuous unitary representations of G with corresponding representation spaces U and V. If there is a $B \in \mathrm{L}(U, V)$ with

[11] A field k is called algebraically closed if every polynomial of degree at least 1 with coefficents in k has a root in k.

$A_B \neq 0$, then $\pi \sim \rho$.

Proof. If $A_B \neq 0$, Schur's lemma tells us that A_B is a bijection. Then Equation (89) tells us A_B can be made into an isometry by multiplication by some $\beta \in \mathbb{C}$ and $\pi \sim \rho$. □

Theorem 95. Let σ_1 and σ_2 be distinct elements in $\digamma$, and let $f_i \in$ be a function in $\mathrm{Trig}_{\sigma_i}(G)$ $(i = 1, 2)$. Then

(i) $\int_G f_1(g)\overline{f}_2(g)dg = 0$.

Proof. Let U_1 and U_2 be the representation spaces for fixed representations $\pi_1 \in \sigma_1$ and $\pi_2 \in \sigma_2$, respectively. Also let d_i denote the dimension of the space U_i and let $\{\xi_{ij}\}_{j=1}^{d_i}$ be a fixed orthonormal basis for U_i. To prove this theorem, we need only show that

$$\int_G \langle \pi_1(g)\xi_{1i}, \xi_{1j}\rangle_1 \overline{\langle \pi_2(g)\xi_{2k}\xi_{2l}\rangle_2} dg = 0$$

for all $1 \leqslant i, j \leqslant d_1$ and $1 \leqslant k, l \leqslant d_2$. The theorem then follows from the fact that linear combinations of functions of the form $\langle \pi_1(g)\xi_{ij}, \xi_{ik}\rangle_1$ make up $\mathrm{Trig}_{\sigma_i}(G)$. Given $1 \leqslant i, j \leqslant d_1$ and $1 \leqslant k, l \leqslant d_2$, we may define a linear transformation $B : U_1 \to U_2$ by $B(\xi_{1j}) = \xi_{2l}$ and $B(\xi_{1m}) = 0$ for $m \neq j$. Then we have that

$$\begin{aligned} \langle B\pi_1(g)\xi_{1i}, \pi_2(g)\xi_{2k}\rangle_2 &= \langle \langle \pi_1(g)\xi_{1i}, \xi_{1j}\rangle_1 \xi_{2l}, \langle \pi_2(g)\xi_{2k}, \xi_{2l}\rangle_2 \xi_{2l}\rangle_2 \\ &= \langle \pi_1(g)\xi_{1i}, \xi_{1j}\rangle_1 \overline{\langle \pi_2(g)\xi_{2k}, \xi_{2l}\rangle_2}. \end{aligned}$$

And therefore

$$\int_G \langle \pi_1(g)\xi_{1i}, \xi_{1j}\rangle_1 \overline{\langle \pi_2(g)\xi_{2k}\xi_{2l}\rangle_2} dx = \int_G \langle B\pi_1(g)\xi_{1i}, \pi_2(g)\xi_{2k}\rangle_2 dg = \langle A_B\xi_{1i}, \xi_{2k}\rangle.$$

By Theorem (94), $\langle A_B\xi_{1i}, \xi_{2k}\rangle = 0$, which proves our theorem. □

Lemma 96. Let π be a continuous irreducible unitary representation of G with d-dimensional representation space U. Let B be any linear operator on U. Then the operator A_B is equal to $d^{-1}\mathrm{tr}(B)I$.

Proof. The operator $A_B \in \mathrm{Twist}(\pi, \pi)$, and therefore is equal to αI for some $\alpha \in \mathbb{C}$. Let $\{\xi_i\}_{i=1}^d$ be an orthonormal basis in U. Then

$$\mathrm{tr}(B) = \mathrm{tr}(\pi(g^{-1})B\pi(g)) = \sum_{i=1}^{d} \langle \pi(g)^{-1}B\pi(g)\xi_i, \xi_i\rangle = \sum_{i=1}^{d} \langle B\pi(g)\xi_i, \pi(g)\xi_i\rangle,$$

so that

$$\mathrm{tr}(B) = \int_G \mathrm{tr}(B)dg = \sum_{i=1}^{d} \int_G \langle B\pi(g)\xi_i, \pi(g)\xi_i\rangle dg = \sum_{i=1}^{d} A_B(\xi_i, \xi_i) = \mathrm{tr}(A_B) = \alpha d$$

and therefore $\alpha = d^{-1}\mathrm{tr}(B)$. □

Theorem 97. Let π be a continuous irreducible unitary representation of G with d-dimensional representation space U. Let $\{\xi_i\}_{i=1}^d$ be an orthonormal basis in U. Then

$$\int_G \langle \pi(g)\xi_i, \xi_j\rangle \overline{\langle \pi(g)\xi_k, \xi_l\rangle} dg = d^{-1}\delta_{ik}\delta_{jl}.$$

Proof. Define a linear operator B_{jl} on U by $B_{jl}(\xi_j) = \xi_l$ and $B_{jl}(\xi_m) = 0$ for $m \neq j$. Then $A_{B_{jl}} = d^{-1}\mathrm{tr}(B_{jl})I = d^{-1}\delta_{jl}I$ and

$$\int_G \langle \pi(g)\xi_i, \xi_j\rangle \overline{\langle \pi(g)\xi_k, \xi_l\rangle} dg = \langle A_{B_{jl}}\xi_i, \xi_k\rangle = d^{-1}\delta_{jl}\langle \xi_i, \xi_k\rangle = d^{-1}\delta_{ik}\delta_{jl}.$$

□

Corollary 98. Let π_1 and π_2 be equivalent continuous irreducible unitary representations of G with corresponding d_i-dimensional representation spaces U_i, and let $T \in \mathrm{Twist}(\pi_1, \pi_2)$ be a linear isometry. Let $\{\xi_{1j}\}_{i=j}^{d_1}$ be an orthonormal basis for U_1

and let $\xi_{2i} = T\xi_{1i}$. Then

$$\int_G \langle \pi_1(g)\xi_{1i}, \xi_{ij}\rangle_1 \overline{\langle \pi_2(g)\xi_{2k}, \xi_{2l}\rangle_2} dg = d^{-1}\delta_{ik}\delta_{jl}.$$

Proof. A quick calculation shows that

$$\langle \pi_2(g)\xi_{2k}, \xi_{2l}\rangle_2 = \langle \pi_2(g)T\xi_{1k}, T\xi_{1l}\rangle_2 = \langle T\pi_1(g)\xi_{1k}, T\xi_{1l}\rangle_1 = \langle \pi_1(g)\xi_{1k}, \xi_{1l}\rangle_1.$$

Therefore the result follows from the previous theorem. □

Gathering up all of our results produces the following orthogonality relations for the coordinate functions

Theorem 99. For each $\sigma \in \Gamma$, let $\{\pi_{jk}^{(\sigma)}\}_{j,k=1}^{d_\sigma}$ be a set of coordinate functions for fixed $\pi_\sigma \in \sigma$ and a fixed basis $\{\xi_i\}_{i=1}^{d_\sigma}$ in the representation space U_σ of π_σ. Then the set of all functions $\sqrt{d_\sigma}\pi_{jk}^{(\sigma)}$ is an orthonormal set in $L_2(G)$. In particular, we have

$$\int_G \pi_{ij}^{(\sigma)}(g)\overline{\pi_{kl}^{(\sigma)}}(g)dg = d_\sigma^{-1}\delta_{ik}\delta_{jl}.$$

Some immediate consequences of this theorem are as follows.

Theorem 100. Let $\sigma, \in \Gamma$, let $\{\pi_{ij}^{(\sigma)}\}_{i,j=1}^{d_\sigma}$ be coordinate functions defined in the usual way for a fixed basis $\{\xi_i\}_{i=1}^{d_\sigma}$ of the representation space U_σ of the representation $\pi_\sigma \in \sigma$, and let $\mu \in \mathbf{M}(G)$.

(i)

$$\mu * \pi_{ij}^{(\sigma)} = \sum_{k=1}^{d_\sigma} \pi_{kj}^{(\sigma)} \int_G \overline{\pi_{ki}^{(\sigma)}(g)} d\mu(g).$$

(ii)

$$\pi_{ij}^{(\sigma)} * \mu = \sum_{k=1}^{d_\sigma} \pi_{ik}^{(\sigma)} \int_G \overline{\pi_{jk}^{(\sigma)}(g)} d\mu(g).$$

(iii) If $\sigma' \in \hat{G}$ and the coordinate functions $\{\pi_{ij}^{(\sigma')}\}_{i,j=1}^{d_\sigma}$ are defined in the usual way for a fixed basis,

$$\pi_{ij}^{(\sigma)} * \pi_{kl}^{(\sigma')} = d_\sigma^{-1} \delta_{\sigma\sigma'} \delta_{jk} \pi_{il}^{(\sigma)},$$

where $\delta_{\sigma\sigma'} = 1$ if σ is equivalent to σ' and 0 otherwise.

(iv) We have the adjointness relation $(\pi_{ij}^{(\sigma)})^* = \pi_{ji}^{(\sigma)}$.

Proof. From the definition of convolution

$$\begin{aligned}(\mu * \pi_{ij}^{(\sigma)})(x) &= \int_G \pi_{ij}^{(\sigma)}(y^{-1}x) d\mu(y) = \sum_{k=1}^{d_\sigma} \int_G \pi_{ik}^{(\sigma)}(y^{-1}) \pi_{kj}^{(\sigma)}(x) d\mu(y) \\ &= \sum_{k=1}^{d_\sigma} \pi_{kj}^{(\sigma)}(x) \int_G \overline{\pi_{ki}^{(\sigma)}(y)} d\mu(y).\end{aligned}$$

Similarly

$$\begin{aligned}(\pi_{ij}^{(\sigma)} * \mu)(x) &= \int_G \pi_{ij}^{(\sigma)}(y) d\mu(y^{-1}x) = \int_G \pi_{ij}^{(\sigma)}(xy^{-1}) d\mu(y) \\ &= \sum_{k=1}^{d_\sigma} \int_G \pi_{ik}^{(\sigma)}(x) \pi_{kj}^{(\sigma)}(y^{-1}) d\mu(y) = \sum_{k=1}^{d_\sigma} \pi_{ik}^{(\sigma)}(x) \int_G \overline{\pi_{jk}^{(\sigma)}(y)} d\mu(y).\end{aligned}$$

From the second identity applied to $d\mu = \pi_{kl} dx$, we obtain from the orthogonality relations

$$\begin{aligned}(\pi_{ij}^{(\sigma)} * \pi_{kl}^{(\sigma)})(x) &= \sum_{m=1}^{d_\sigma} \pi_{im}^{(\sigma)}(x) \int_G \overline{\pi_{jm}^{(\sigma)}(y)} \pi_{kl}^{(\sigma)}(y) dy = \frac{1}{d_\sigma} \sum_{m=1}^{d_\sigma} \pi_{im}^{(\sigma)}(x) \delta_{jk} \delta_{ml} \\ &= \frac{1}{d_\sigma} \pi_{il}^{(\sigma)}(x) \delta_{jk}.\end{aligned}$$

From this (iii) is obvious. Lastly, we have that

$$(\pi_{ij}^{(\sigma)})^*(g) = \overline{\pi_{ij}^{(\sigma)}(g^{-1})} = \overline{\langle \pi_\sigma(g^{-1})\xi_i, \xi_j \rangle} = \overline{\langle \xi_i, \pi_\sigma(g)\xi_j \rangle} = \langle \pi_\sigma(g)\xi_j, \xi_i \rangle = \pi_{ji}^{(\sigma)}(g).$$

□

We now have the toolkit (or utility belt, if the reader prefers) to address the question left over from the previous section. That is, we may show that for $\mathrm{Trig}_{\mathfrak{p}}(G)$ to be dense in $C(G)$, we must have that $\mathfrak{p} = \widehat{G}$. This is established by the next theorem.

Theorem 101. Let $\mathfrak{p} \subset \widehat{G}$ be closed under tensor products and conjugation. Then the following are equivalent:

(i) $\mathfrak{p} = \widehat{G}$;

(ii) $\mathrm{Trig}_{\mathfrak{p}}(G)$ is dense in $C(G)$;

(iii) $\mathfrak{p}$ separates points in G.

Proof. The proof of (i) implies (iii) and (iii) implies (ii) were already taken care of in the previous section. We need now only show that (ii) implies (i). Suppose that $\mathfrak{p} \neq \widehat{G}$. Then we may choose $\sigma \notin \mathfrak{p}$ with $\sigma \in \widehat{G}$. Let $f \in \mathrm{Trig}_{\sigma}(G)$ with $\|f\|_2^2 \neq 1$. If $\mathrm{Trig}_{\mathfrak{p}}(G)$ is dense in $C(G)$, there exists a $g \in \mathrm{Trig}_{\mathfrak{p}}(G)$ with $\|f - g\|_2^2 < \|f\|_2^2$. However, since $\sigma \notin \mathfrak{p}$ we know that f and g are orthogonal and therefore $\|f - g\|_2^2 = \|f\|_2^2 + \|g\|_2^2 \geqslant \|f\|_2^2$. This is a contradiction, and we conclude that (ii) implies (i). □

The orthogonality results and density results we have thus far attained are summarized in the Peter-Weyl theorem.

Theorem 102 (Peter-Weyl)**.** For each $\sigma \in \widehat{G}$, fix a representation $\pi_\sigma \in \sigma$ with representation space H_σ of dimension d_σ. Also fix a basis $\{\xi_i^{(\sigma)}\}_{i=1}^{d_\sigma}$ of H_σ and define the coordinate functions $\pi_{ij}^{(\sigma)}(x) = \langle \pi_\sigma(x)\xi_i^{(\sigma)}, \xi_j^{(\sigma)} \rangle$ as usual. The collection of functions $\{d_\sigma^{1/2}\pi_{ij}^{(\sigma)} : \sigma \in \widehat{G}, 1 \leqslant i, j \leqslant d_\sigma\}$ forms an orthonormal basis for $L_2(G)$.

Proof. The fact that the collection is orthonormal follows immediately from the orthogonality relations. The fac that the collection is dense in $L_2(G)$ follows from the fact that it is dense in $C(G)$ and $C(G)$ is dense in $L_2(G)$. $\square$

3.5. The Fourier-Stieltjes Transform

For this entire section, for each $\sigma \in \hat{G}$, fix a representation $\pi_\sigma \in \sigma$ with representation space H_σ of dimension d_σ. Also fix a basis $\{\xi_i^{(\sigma)}\}_{i=1}^{d_\sigma}$ of H_σ and define the coordinate functions $\pi_{ij}^{(\sigma)}(x) = \langle \pi_\sigma(x)\xi_i^{(\sigma)}, \xi_j^{(\sigma)} \rangle$ as usual. Using this convention, we define the Fourier-Stieltjes transform of a measure $\mu \in \mathbf{M}(G)$.

Definition 103. Let $\mu \in \mathbf{M}(G)$. For each $\sigma \in \hat{G}$, we define $\hat{\mu}(\sigma)$ to be the operator on H_σ defined by

$$\langle \hat{\mu}(\sigma)\psi, \phi \rangle = \sqrt{d_\sigma} \int_G \overline{\langle \pi_\sigma(x)\psi, \phi \rangle} d\mu(x). \tag{27}$$

The function $\hat{\mu}$ is called the *Fourier-Stieltjes transform* of μ. For $f \in L_1(G)$, we define $\hat{f} = \hat{\mu}$, where $d\mu = f dx$.

From the definition, we have the linearity properties $\widehat{\mu + \nu} = \hat{\mu} + \hat{\nu}$ and $\widehat{\alpha\mu} = \alpha\hat{\mu}$ for all $\mu, \nu \in \mathbf{M}(G)$ and $\alpha \in \mathbb{C}$. By the orthogonality relations, we see that

$$\begin{aligned}
\left\langle \widehat{\pi_{ij}^{(\sigma)}}(\sigma')\psi, \phi \right\rangle &= \sqrt{d_\sigma} \int_G \langle \pi_{\sigma'}(x)\psi, \phi \rangle \pi_{ij}^{(\sigma)}(x) dx \\
&= \sqrt{d_\sigma} \sum_{k,l=1}^{d_{\sigma'}} \langle \psi, \xi_k \rangle \langle \xi_l, \phi \rangle \int_G \pi_{kl}^{(\sigma')}(x) \pi_{ij}^{(\sigma)}(x) dx \\
&= \frac{1}{\sqrt{d_\sigma}} \langle \psi, \xi_k \rangle \langle \xi_l, \phi \rangle \delta_{ki} \delta_{lj} \delta_{\sigma\sigma'} = \frac{1}{\sqrt{d_\sigma}} \langle \psi, \xi_i \rangle \langle \xi_j, \phi \rangle \delta_{\sigma\sigma'}.
\end{aligned}$$

By the Peter-Weyl theorem, for any $f \in L_2(G)$ we have an expansion

$$f(x) = \sum_{\sigma \in \hat{G}} \sum_{i,j=1}^{d_\sigma} d_\sigma^{1/2} f_{ij}^{(\sigma)} \pi_{ij}^{(\sigma)}(x),$$

where $f_{ij}^{(\sigma)} \in \mathbb{C}$. It follows that

$$\langle \hat{f}(\sigma')\psi, \phi\rangle = \sum_{\sigma \in Ᵽ} \sum_{i,j=1}^{d_\sigma} f_{ij}^{(\sigma)} \langle \psi, \xi_i\rangle \langle \xi_j, \phi\rangle \delta_{\sigma\sigma'},$$

so that in particular, the matrix components of $\hat{f}(\sigma)$ are

$$\langle \hat{f}(\sigma)\xi_i, \xi_j\rangle = f_{ij}^{(\sigma)}.$$

Example 104. As an example of finding the Fourier-Stieltjes transform of a function f, we will consider the case when the group G has finite cardinality. In this case, we may enumerate the members of G by $G = \{g_i\}_{i=1}^n$, with g_1 being the identity for convenience. The group G is a compact topological group under the discrete topology[12]and every complex-valued function on G is continuous. The (normalized) Haar measure on G is defined by $\lambda(S) = \frac{1}{n}|S|$ for any arbitrary subset $S \subset G$, where $|S|$ denotes the number of elements contained in S. It follows that any complex-valued function on G is in $L_2(G)$, and that $L_2(G)$ is isometrically isomorphic with $\mathbb{C}^n$ as vector spaces under the correspondence $f \in L_2(G)$ maps to a vector $x \in \mathbb{C}^n$, whose i'th coordinate x_i is given by $x_i = f(g_i)$ for all $1 \leqslant i \leqslant n$. For sake of convenience, we will use $f \in L_2(G)$ to denote both the function on G and the vector in $L_2(G)$, with i'th coordinate $f_i = f(g_i)$.

Now consider Ᵽ, and for each $\sigma \in Ᵽ$ fix a representation $\pi_\sigma \in \sigma$ and a basis $\{\xi_i^{(\sigma)}\}_{i=1}^{d_\sigma}$ for the representation space of π_σ. Additionally, we again let $\pi_{ij}^{(\sigma)}$ denote the coordinate map defined by $\pi_{ij}^{(\sigma)}(g) = \langle \pi_\sigma(g)\xi_i^{(\sigma)}, \xi_j^{(\sigma)}\rangle$. Then for the same reasons as above, $\pi_{ij}^{(\sigma)}$ is a vector in $\mathbb{C}^n$ for all $1 \leqslant i, j \leqslant d_\sigma$ with k'th coordinate denoted by $\pi_{ijk}^{(\sigma)} = \pi_{ij}^{(\sigma)}(g_k)$. Recall that $\{\pi_{ij}^{(\sigma)} : \sigma \in Ᵽ,\ 1 \leqslant i, j \leqslant d_\sigma\}$ is an orthonormal basis for

[12] Let X be a set. The discrete topology for X is the collection of all subsets of X, and with this topology X is a topological space.

$L_2(G)$, and therefore for $\mathbb{C}^n$. This tells us that the number of elements in the basis must be n, or rather $\sum_{\sigma \in \hat{G}} d_\sigma^2 = n$. Thus there are finitely many elements in the dual of G, and we enumerate them as $\hat{G} = \{\sigma_i\}_{i=1}^r$. Since there will in general be n coordinate maps on G, each of which is representable as a vector in $\mathbb{C}^n$, the Fourier-Stieltjes transform will in general be able to be represented as an $n \times n$ invertible matrix F, the rows of which are the coordinate maps.

As a specific example of this, consider the case when $G = S_3$. We index the elements of G[13] in the following way:

Element	Index
e	1
(12)	2
(13)	3
(23)	4
(123)	5
(132)	6

Them multiplication table for this group is then

	$\mathbf{g_1}$	$\mathbf{g_2}$	$\mathbf{g_3}$	$\mathbf{g_4}$	$\mathbf{g_5}$	$\mathbf{g_6}$
$\mathbf{g_1}$	g_1	g_2	g_3	g_4	g_5	g_6
$\mathbf{g_2}$	g_2	g_1	g_5	g_6	g_4	g_3
$\mathbf{g_3}$	g_3	g_6	g_1	g_5	g_2	g_4
$\mathbf{g_4}$	g_4	g_5	g_6	g_1	g_3	g_2
$\mathbf{g_5}$	g_5	g_3	g_4	g_2	g_6	g_1
$\mathbf{g_6}$	g_6	g_4	g_2	g_3	g_1	g_5

Let π be a representation of G. We start by assuming that the dimension of the representation space of π is 1, i.e. that π is a homomorphism of G into $\mathbb{C}$. Then

[13]We denote the elements of G in the usual cycle notation.

necessarily $\pi(g_1) = 1$, and since $g_5^3 = 1$, we must have that $\pi(g_5) \in \{\omega, \omega^2, 1\}$, where $\omega = \exp(2\pi i/3)$. Also, $g_2^2 = g_3^2 = g_4^4 = e$, and therefore $\pi(g_2), \pi(g_3), \pi(g_4) \in \{-1, 1\}$. Moreover, $g_4 g_2 = g_2 g_3 = g_3 g_4 = g_5$, and it follows immediately that $\pi(g_2) = \pi(g_3) = \pi(g_4)$ and therefore $\pi(g_5) = \pi(g_2)^2 = 1$. Thus we find exactly two irreducible unitary representations of dimension 1: the mapping $1 : g_i \mapsto 1$ for all $1 \leqslant i \leqslant 6$ and the mapping

$$\rho(g_i) = \begin{cases} -1, & i \in \{2, 3, 4\}; \\ 1, & \text{otherwise.} \end{cases}$$

Next, we assume that the dimension of the representation space of π is 2, and see if we can arrive at any new representations that are not reducible into one of the previously found representations. It may be shown that the representation σ defined by

$$g_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad g_2 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad g_3 \mapsto \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$
$$g_4 \mapsto \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} g_5 \mapsto \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix} \quad g_6 \mapsto \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$$

This must be all the irreducible unitary representations, since we have two dimension 1 representations and 1 dimension 2 representation and $1^2 + 1^2 + 2^2 = 6 = |G|$. We conclude that $\hat{G} = \{1, \rho, \overline{\sigma}\}$, where $\overline{\sigma}$ represents the equivalence class of the representation σ defined above. The Fourier transform matrix F is therefore given by

$$F = \frac{1}{|G|}\overline{\begin{pmatrix} 1(g_1) & 1(g_2) & 1(g_3) & 1(g_4) & 1(g_5) & 1(g_6) \\ \rho(g_1) & \rho(g_2) & \rho(g_3) & \rho(g_4) & \rho(g_5) & \rho(g_6) \\ \sqrt{2}\sigma(g_1)_{11} & \sqrt{2}\sigma(g_2)_{11} & \sqrt{2}\sigma(g_3)_{11} & \sqrt{2}\sigma(g_4)_{11} & \sqrt{2}\sigma(g_5)_{11} & \sqrt{2}\sigma(g_6)_{11} \\ \sqrt{2}\sigma(g_1)_{12} & \sqrt{2}\sigma(g_2)_{12} & \sqrt{2}\sigma(g_3)_{12} & \sqrt{2}\sigma(g_4)_{12} & \sqrt{2}\sigma(g_5)_{12} & \sqrt{2}\sigma(g_6)_{12} \\ \sqrt{2}\sigma(g_1)_{21} & \sqrt{2}\sigma(g_2)_{21} & \sqrt{2}\sigma(g_3)_{21} & \sqrt{2}\sigma(g_4)_{21} & \sqrt{2}\sigma(g_5)_{21} & \sqrt{2}\sigma(g_6)_{21} \\ \sqrt{2}\sigma(g_1)_{22} & \sqrt{2}\sigma(g_2)_{22} & \sqrt{2}\sigma(g_3)_{22} & \sqrt{2}\sigma(g_4)_{22} & \sqrt{2}\sigma(g_5)_{22} & \sqrt{2}\sigma(g_6)_{22} \\ \sqrt{2} & & & & & \end{pmatrix}}$$

$$= \frac{1}{6}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & 1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2}/2 & \sqrt{2}/2 & -\sqrt{2}/2 & -\sqrt{2}/2 \\ 0 & 0 & -\sqrt{6}/2 & \sqrt{6}/2 & -\sqrt{6}/2 & \sqrt{6}/2 \\ 0 & 0 & -\sqrt{6}/2 & \sqrt{6}/2 & \sqrt{6}/2 & -\sqrt{6}/2 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2}/2 & -\sqrt{2}/2 & -\sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix}.$$

The matrix components of the Fourier-Stieltjes transform of f in vector form are then given by the matrix product $\hat{f} = Ff$.

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